A Concise Summary of @RISK Probability Distribution Functions

Beta

RISKBeta(α1, α2)

Parameters:

α_1	continuous shape parameter	$\alpha_1 > 0$
α_2	continuous shape parameter	$\alpha_2 > 0$

Domain:

$0 \le x \le 1$ continuous

Density and Cumulative Distribution Functions:

$$f(x) = \frac{x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)}$$

$$F(x) = \frac{B_x(\alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \equiv I_x(\alpha_1, \alpha_2)$$

where B is the *Beta Function* and B_x is the *Incomplete Beta Function*.

Mean:

$$\frac{\alpha_1}{\alpha_1 + \alpha_2}$$

Variance:

$$\frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2(\alpha_1 + \alpha_2 + 1)}$$

Skewness:

$$2\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2 + 2}\sqrt{\frac{\alpha_1 + \alpha_2 + 1}{\alpha_1\alpha_2}}$$

Kurtosis:

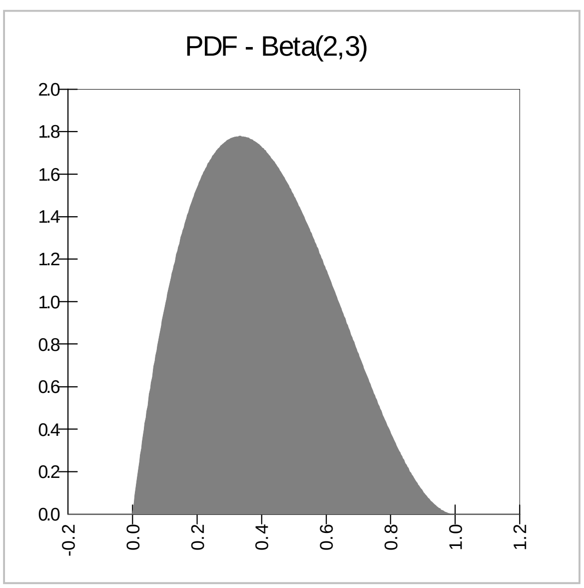
$$3\frac{(\alpha_1 + \alpha_2 + 1)\left(2(\alpha_1 + \alpha_2)^2 + \alpha_1\alpha_2(\alpha_1 + \alpha_2 - 6)\right)}{\alpha_1\alpha_2(\alpha_1 + \alpha_2 + 2)(\alpha_1 + \alpha_2 + 3)}$$

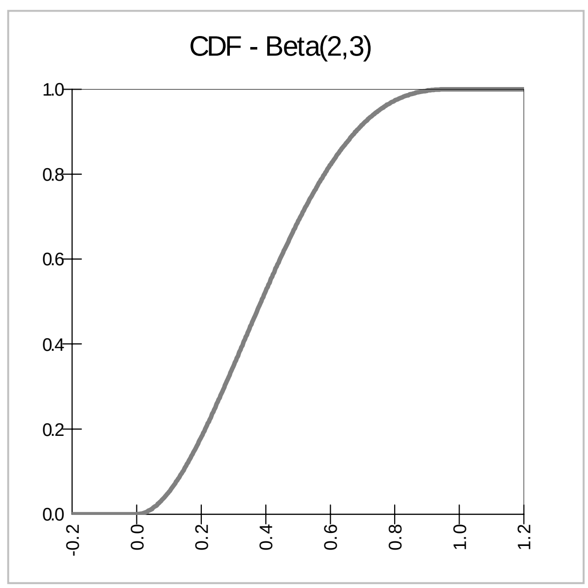
Mode:

$\frac{\alpha_1 - 1}{\alpha_1 + \alpha_2 - 2}$ $\alpha_1>1,\ \alpha_2>1$

0 $\alpha_1<1,\ \alpha_2\geq 1$ or $\alpha_1=1,\ \alpha_2>1$

1 $\alpha_1\geq 1,\ \alpha_2<1$ or $\alpha_1>1,\ \alpha_2=1$





Beta (Generalized)

RISKBetaGeneral(α1, α2, min, max)

Parameters:

α_1	continuous shape parameter	$\alpha_1 > 0$
α_2	continuous shape parameter	$\alpha_2 > 0$
min	continuous boundary parameter	min < max
max	continuous boundary parameter	

Domain:

$\min \le x \le \max$ continuous

Density and Cumulative Distribution Functions:

$$f(x) = \frac{(x-\min)^{\alpha_1 - 1}(\max - x)^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)(\max - \min)^{\alpha_1 + \alpha_2 - 1}}$$

$$F(x) = \frac{B_z(\alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \equiv I_z(\alpha_1, \alpha_2) \qquad \text{with } z \equiv \frac{x - \min}{\max - \min}$$

where B is the *Beta Function* and B_z is the *Incomplete Beta Function.*

Mean:

$$\min + \frac{\alpha_1}{\alpha_1 + \alpha_2}(\max - \min)$$

Variance:

$$\frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2(\alpha_1 + \alpha_2 + 1)}(\max - \min)^2$$

Skewness:

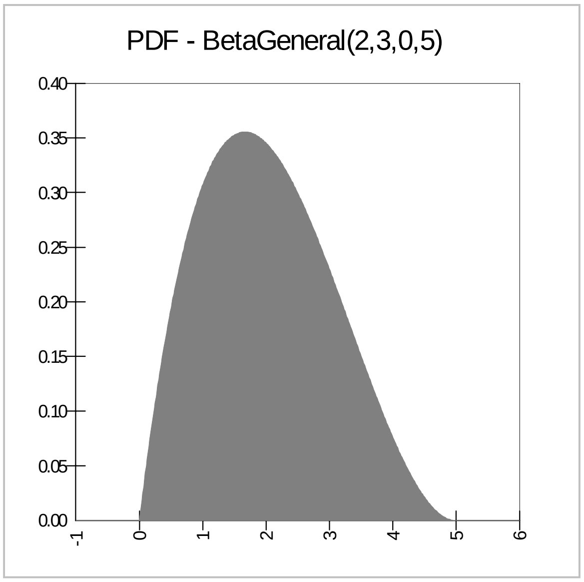
$$2\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2+2}\sqrt{\frac{\alpha_1+\alpha_2+1}{\alpha_1\alpha_2}}$$

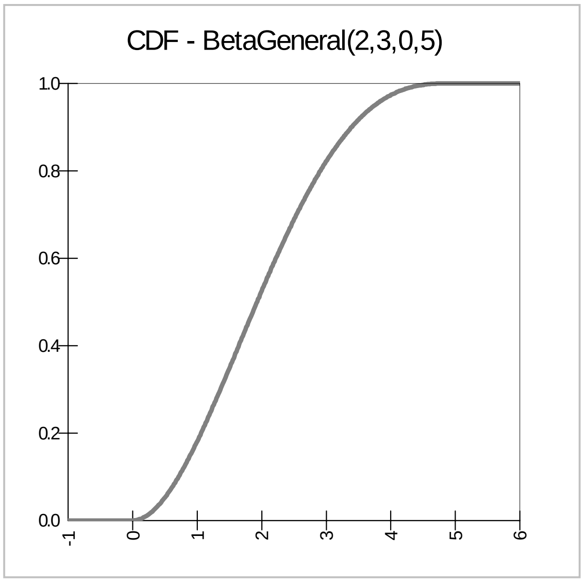
Kurtosis:

$$3\frac{(\alpha_1+\alpha_2+1)\left(2(\alpha_1+\alpha_2)^2+\alpha_1\alpha_2(\alpha_1+\alpha_2-6)\right)}{\alpha_1\alpha_2(\alpha_1+\alpha_2+2)(\alpha_1+\alpha_2+3)}$$

Mode:

$\min+\frac{\alpha_1-1}{\alpha_1+\alpha_2-2}(\max-\min)$	$\alpha_1>1$, $\alpha_2>1$
min	$\alpha_1<1$, $\alpha_2\geq1$ or $\alpha_1=1$, $\alpha_2>1$
max	$\alpha_1\geq1$, $\alpha_2<1$ or $\alpha_1>1$, $\alpha_2=1$





Beta (Subjective)

RISKBetaSubj(min, m.likely, mean, max)

Definitions:

$$\text{mid} \equiv \frac{\text{min} + \text{max}}{2}$$

$$\alpha_1 \equiv 2\frac{(\text{mean} - \text{min})(\text{mid} - \text{m.likely})}{(\text{mean} - \text{m.likely})(\text{max} - \text{min})} \qquad \alpha_2 \equiv \alpha_1 \frac{\text{max} - \text{mean}}{\text{mean} - \text{min}}$$

Parameters:

min	continuous boundary parameter	min < max
m.likely	continuous parameter	min < m.likely < max
mean	continuous parameter	min < mean < max
max	continuous boundary parameter	
	mean > mid	if m.likely > mean
	mean < mid	if m.likely < mean
	mean = mid	if m.likely = mean

Domain:

$\text{min} \le x \le \text{max}$ continuous

Density and Cumulative Distribution Functions:

$$f(x) = \frac{(x - \text{min})^{\alpha_1 - 1}(\text{max} - x)^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)(\text{max} - \text{min})^{\alpha_1 + \alpha_2 - 1}}$$

$$F(x) = \frac{B_z(\alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \equiv I_z(\alpha_1, \alpha_2) \qquad \text{with} \quad z \equiv \frac{x - \text{min}}{\text{max} - \text{min}}$$

where B is the *Beta Function* and B_z is the *Incomplete Beta Function*.

Mean:

$$\text{mean}$$

Variance:

$$\frac{(\text{mean}-\text{min})(\text{max}-\text{mean})(\text{mean}-\text{m.likely})}{2\cdot\text{mid}+\text{mean}-3\cdot\text{m.likely}}$$

Skewness:

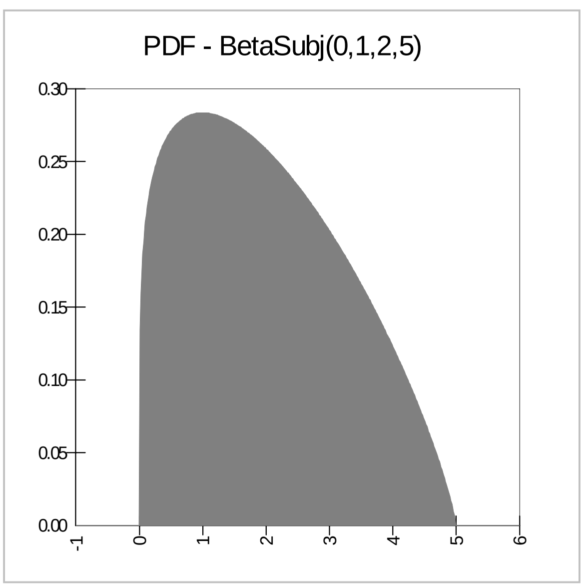
$$\frac{2(\text{mid}-\text{mean})}{|\text{mean}+\text{mid}-2\cdot\text{m.likely}|}\sqrt{\frac{(\text{mean}-\text{m.likely})(2\cdot\text{mid}+\text{mean}-3\cdot\text{m.likely})}{(\text{mean}-\text{min})(\text{max}-\text{mean})}}$$

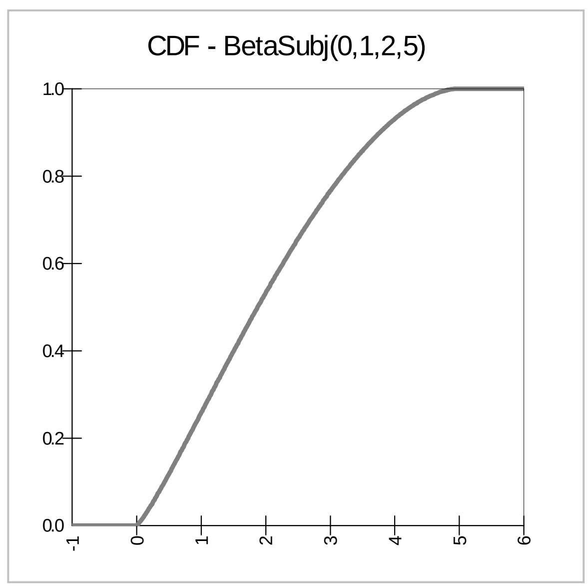
Kurtosis:

$$3\frac{(\alpha_1+\alpha_2+1)\left(2(\alpha_1+\alpha_2)^2+\alpha_1\alpha_2(\alpha_1+\alpha_2-6)\right)}{\alpha_1\alpha_2(\alpha_1+\alpha_2+2)(\alpha_1+\alpha_2+3)}$$

Mode:

$$\text{m.likely}$$





Binomial

RISKBinomial(n, p)

Parameters:

n	discrete "count" parameter	$n > 0$
p	continuous "success" probability	$0 < p < 1$

Domain:

$0 \le x \le n$ discrete

Mass and Cumulative Functions:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$F(x) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$$

Mean:

np

Variance:

$np(1-p)$

Skewness:

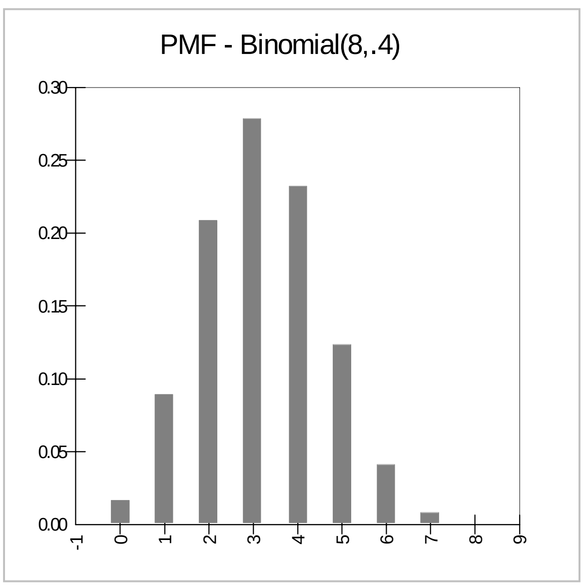
$$\frac{(1-2p)}{\sqrt{np(1-p)}}$$

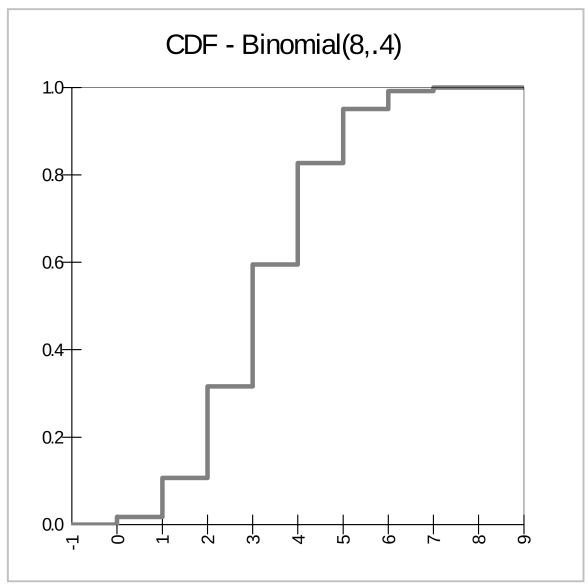
Kurtosis:

$$3-\frac{6}{n}+\frac{1}{np(1-p)}$$

Mode:

(bimodal)	$p(n+1)-1$ and $p(n+1)$	if $p(n+1)$ is integral
(unimodal)	largest integer less than $p(n+1)$	otherwise





Chi-Squared

RISKChiSq(ν)

Parameters:

ν	discrete shape parameter	$\nu > 0$

Domain:

$0 \le x \le +\infty$	continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2} x^{(\nu/2)-1}$$

$$F(x) = \frac{\Gamma_{x/2}(\nu/2)}{\Gamma(\nu/2)}$$

where Γ is the *Gamma Function*, and Γ_x is the *Incomplete Gamma Function*.

Mean:

ν

Variance:

2ν

Skewness:

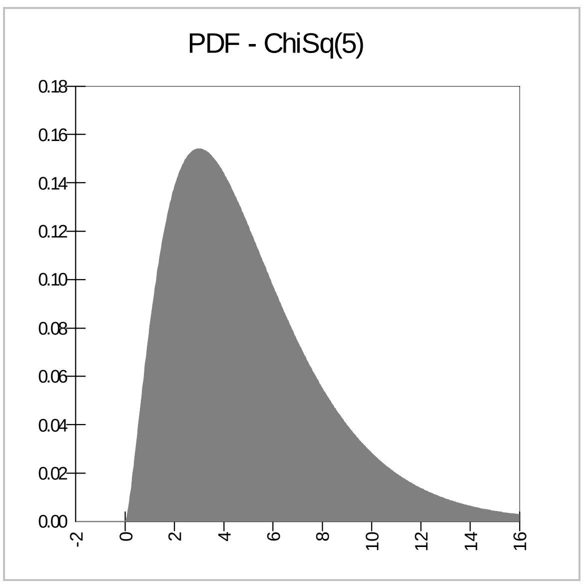
$$\sqrt{\frac{8}{\nu}}$$

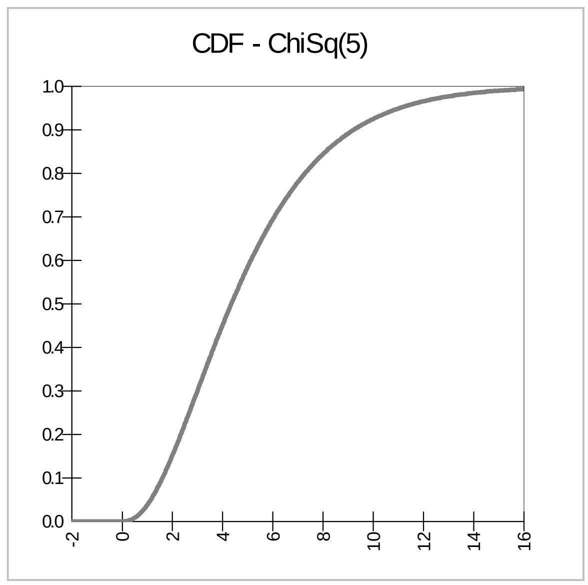
Kurtosis:

$$3+\frac{12}{\nu}$$

Mode:

ν-2	if $\nu \geq 2$
0	if $\nu = 1$





Cumulative (Ascending)

RISKCumul(min, max, {x}, {p})

Parameters:

min	continuous parameter	min < max
max	continuous parameter	
$\{x\} = \{x_1, x_2, \ldots, x_N\}$	array of continuous parameters	$min \le x_i \le max$
$\{p\} = \{p_1, p_2, \ldots, p_N\}$	array of continuous parameters	$0 \le p_i \le 1$

Domain:

$min \le x \le max$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{p_{i+1} - p_i}{x_{i+1} - x_i} \qquad \text{for } x_i \le x < x_{i+1}$$

$$F(x) = p_i + (p_{i+1} - p_i)\left(\frac{x - x_i}{x_{i+1} - x_i}\right) \qquad \text{for } x_i \le x \le x_{i+1}$$

With the assumptions:

1. The arrays are ordered from left to right
2. The *i* index runs from 0 to N+1, with two extra elements : $x_0 \equiv min$, $p_0 \equiv 0$ and $x_{N+1} \equiv max$, $p_{N+1} \equiv 1$.

Mean:

No Closed Form

Variance:

No Closed Form

Skewness:

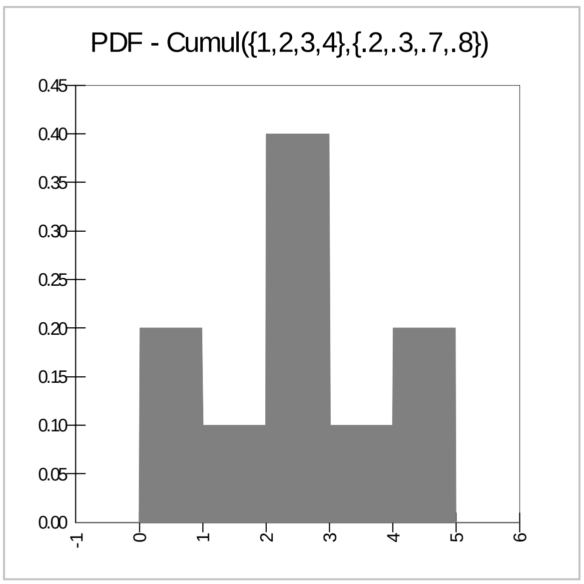
No Closed Form

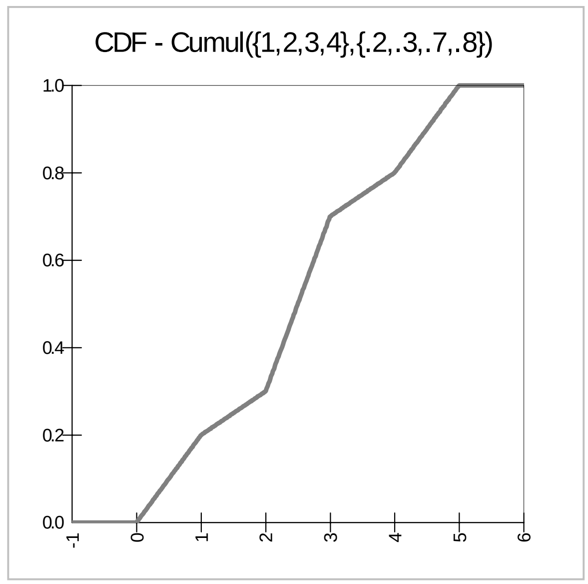
Kurtosis:

No Closed Form

Mode:

No Closed Form





Cumulative (Descending)

RISKCumulD(min, max, {x}, {p})

Parameters:

min	continuous parameter	min < max
max	continuous parameter	
$\{x\} = \{x_1, x_2, \ldots, x_N\}$	array of continuous parameters	$min \le x_i \le max$
$\{p\} = \{p_1, p_2, \ldots, p_N\}$	array of continuous parameters	$0 \le p_i \le 1$

Domain:

$min \le x \le max$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{p_i - p_{i+1}}{x_{i+1} - x_i} \quad \text{for } x_i \le x < x_{i+1}$$

$$F(x) = 1 - p_i + (p_i - p_{i+1})\left(\frac{x - x_i}{x_{i+1} - x_i}\right) \quad \text{for } x_i \le x \le x_{i+1}$$

With the assumptions:

1. The arrays are ordered from left to right
2. The *i* index runs from 0 to N+1, with two extra elements : $x_0 \equiv min$, $p_0 \equiv 1$ and $x_{N+1} \equiv max$, $p_{N+1} \equiv 0$.

Mean:

No Closed Form

Variance:

No Closed Form

Skewness:

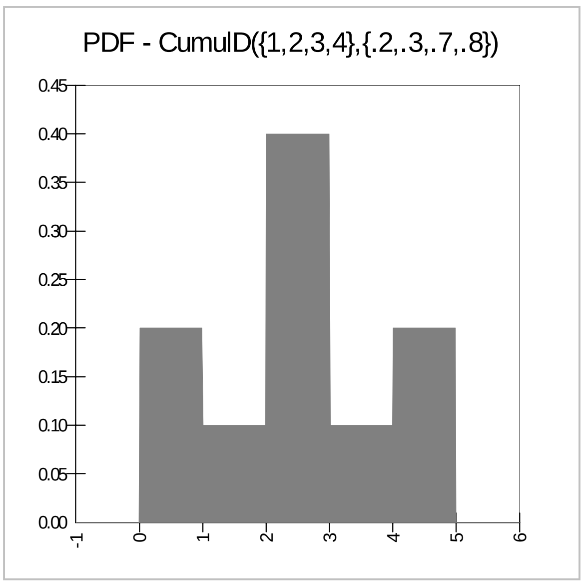
No Closed Form

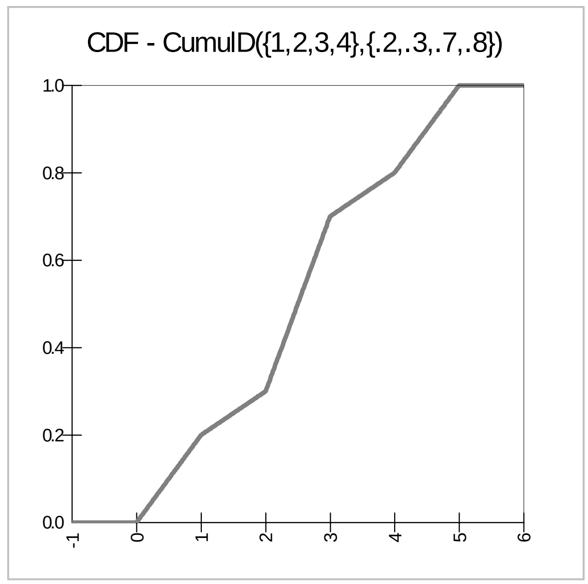
Kurtosis:

No Closed Form

Mode:

No Closed Form





Discrete

RISKDiscrete({x}, {p})

Parameters:

$\{x\} = \{x_1, x_2, \ldots, x_N\}$ array of continuous parameters

$\{p\} = \{p_1, p_2, \ldots, p_N\}$ array of continuous parameters

Domain:

$x \in \{x\}$ discrete

Mass and Cumulative Functions:

$f(x) = p_i$ for $x = x_i$

$f(x) = 0$ for $x \notin \{x\}$

$F(x) = 0$ for $x < x_1$

$F(x) = \sum_{i=1}^{s} p_i$ for $x_s \leq x < x_{s+1}$, $s < N$

$F(x) = 1$ for $x \geq x_N$

With the assumptions:

1. The arrays are ordered from left to right
2. The p array is normalized to 1.

Mean:

$$\sum_{i=1}^{N} x_i p_i \equiv \mu$$

Variance:

$$\sum_{i=1}^{N}(x_i - \mu)^2 p_i \equiv V$$

Skewness:

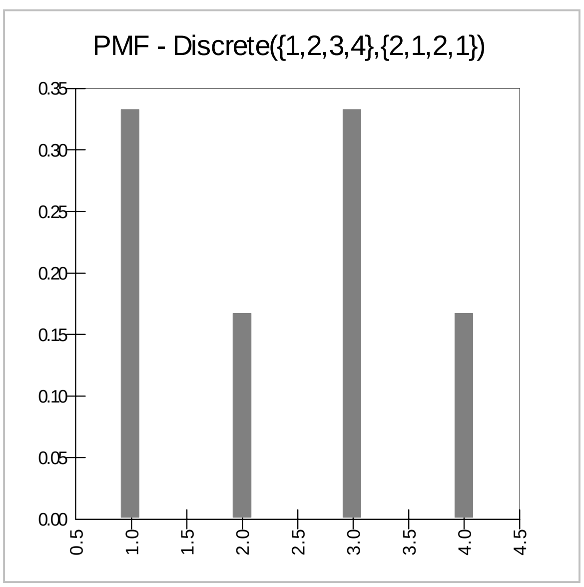
$$\frac{1}{V^{3/2}}\sum_{i=1}^{N}(x_i - \mu)^3 p_i$$

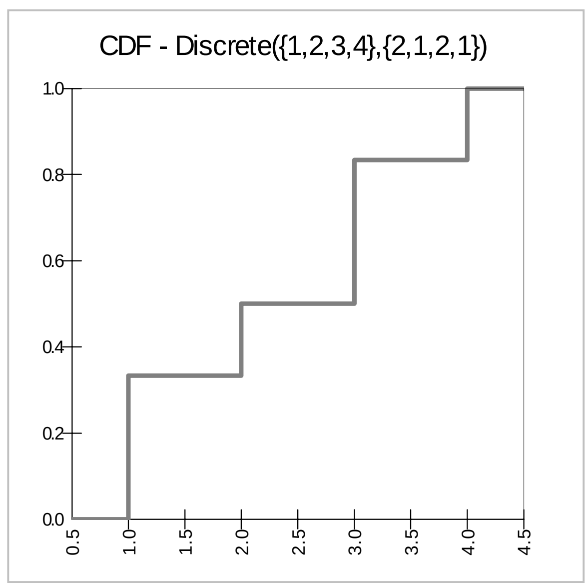
Kurtosis:

$$\frac{1}{V^2}\sum_{i=1}^{N}(x_i - \mu)^4 p_i$$

Mode:

The x-value corresponding to the highest p-value.





Discrete Uniform

RISKDUniform({x})

Parameters:

$\{x\} = \{x_1, x_1, \ldots, x_N\}$ array of continuous parameters

Domain:

$x \in \{x\}$ discrete

Mass and Cumulative Functions:

$$f(x) = \frac{1}{N} \quad \text{for } x \in \{x\}$$

$$f(x) = 0 \quad \text{for } x \notin \{x\}$$

$$F(x) = 0 \quad \text{for } x < x_1$$

$$F(x) = \frac{i}{N} \quad \text{for } x_i \le x < x_{i+1}$$

$$F(x) = 1 \quad \text{for } x \ge x_N$$

assuming the {x} array is ordered.

Mean:

$$\frac{1}{N}\sum_{i=1}^{N} x_i \equiv \mu$$

Variance:

$$\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2 \equiv V$$

Skewness:

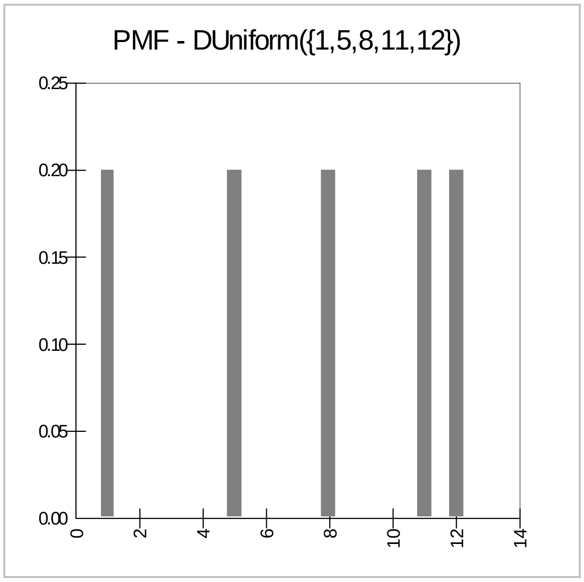
$$\frac{1}{NV^{3/2}}\sum_{i=1}^{N}(x_i-\mu)^3$$

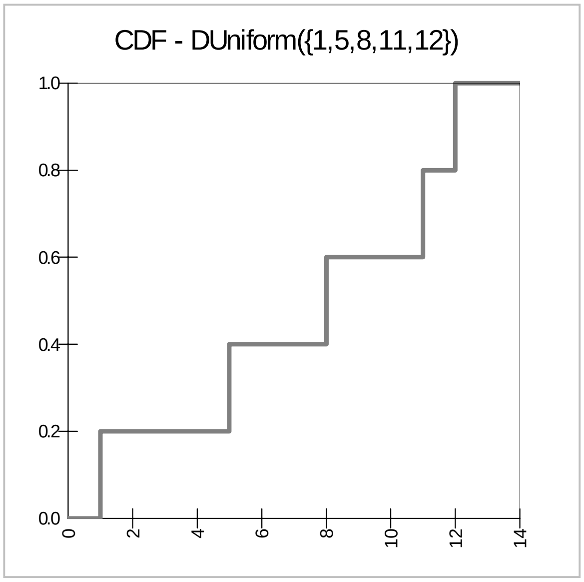
Kurtosis:

$$\frac{1}{NV^2}\sum_{i=1}^{N}(x_i-\mu)^4$$

Mode:

Not uniquely defined





“Error Function”

RISKErf(h)

<u>Parameters</u>:

h	continuous inverse scale parameter	$h > 0$

<u>Domain</u>:

$-\infty \le x \le +\infty$	continuous

<u>Density and Cumulative Functions</u>:

$$f(x) = \frac{h}{\sqrt{\pi}} e^{-(hx)^2}$$

$$F(x) = \Phi\left(\sqrt{2}hx\right)$$

where Φ is the *Error Function*.

<u>Mean</u>:

0

<u>Variance</u>:

$$\frac{1}{2h^2}$$

Skewness:

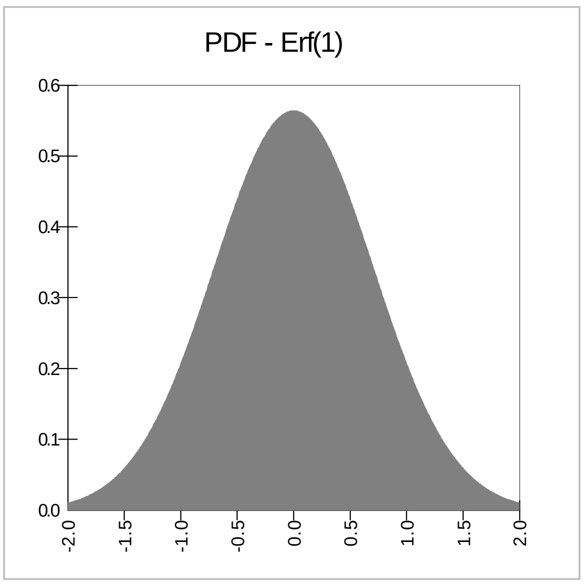
0

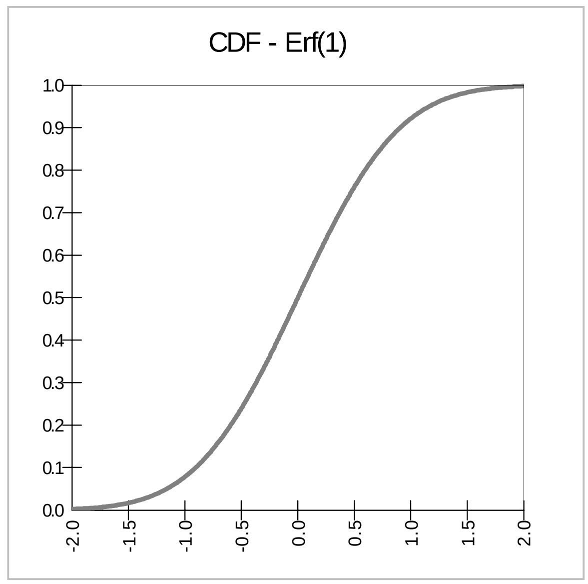
Kurtosis:

3

Mode:

0





Erlang

RISKErlang(m,β)

Parameters:

m	integral shape parameter	$m > 0$
β	continuous scale parameter	$\beta > 0$

Domain:

$0 \le x < +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{\beta\,(m-1)!}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta}$$

$$F(x) = \frac{\Gamma_{x/\beta}(m)}{\Gamma(m)} = 1 - e^{-x/\beta}\sum_{i=0}^{m-1}\frac{(x/\beta)^i}{i!}$$

where Γ is the *Gamma Function* and Γ_x is the *Incomplete Gamma Function*.

Mean:

$m\beta$

Variance:

$m\beta^2$

Skewness:

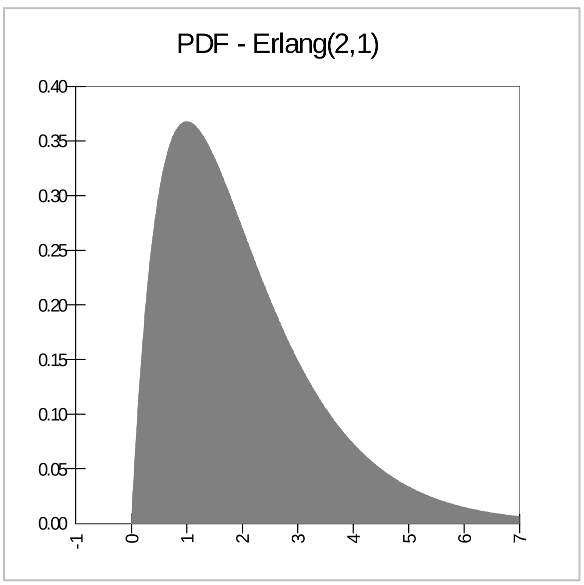
$$\frac{2}{\sqrt{m}}$$

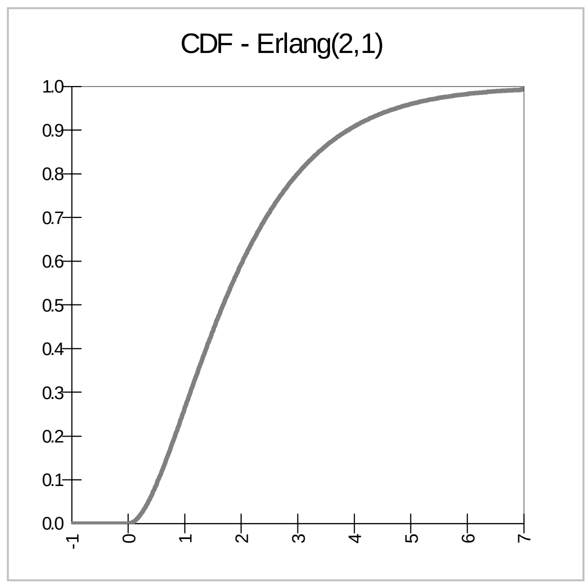
Kurtosis:

$$3 + \frac{6}{m}$$

Mode:

$$\beta(m-1)$$





Exponential

RISKExpon(β)

<u>Parameters</u>:

β	continuous scale parameter	$\beta > 0$

<u>Domain</u>:

$0 \le x < +\infty$	continuous

<u>Density and Cumulative Functions</u>:

$$f(x) = \frac{e^{-x/\beta}}{\beta}$$

$$F(x) = 1 - e^{-x/\beta}$$

<u>Mean</u>:

β

<u>Variance</u>:

β^2

Skewness:

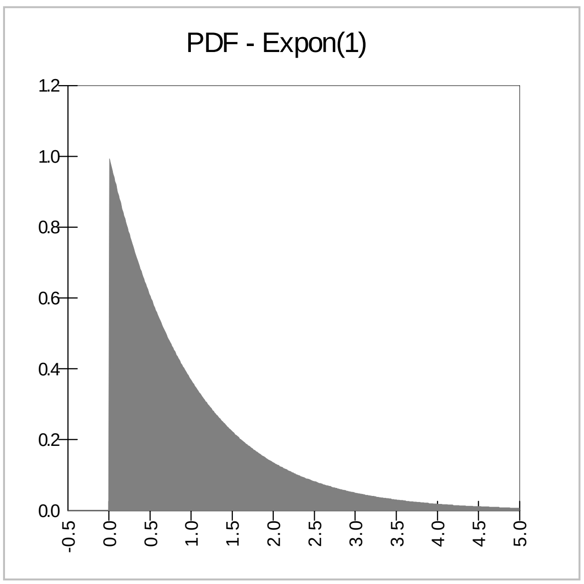
2

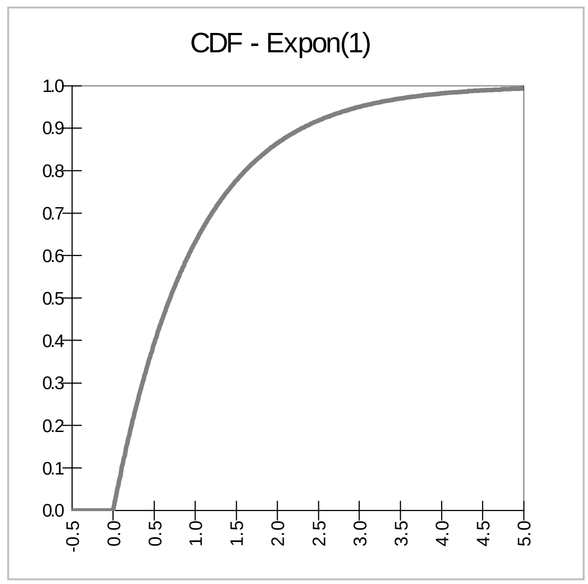
Kurtosis:

9

Mode:

0





Extreme Value

RISKExtValue(a, b)

Parameters:

a	continuous location parameter	
b	continuous scale parameter	$b > 0$

Domain:

$-\infty \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{b}\left(\frac{1}{e^{z+\exp(-z)}}\right)$$

$$F(x) = \frac{1}{e^{\exp(-z)}}$$

where $z \equiv \frac{(x-a)}{b}$

Mean:

$$a - b\Gamma'(1) \approx a + .577b$$

where Γ'(x) is the derivative of the *Gamma Function*.

Variance:

$$\frac{\pi^2 b^2}{6}$$

Skewness:

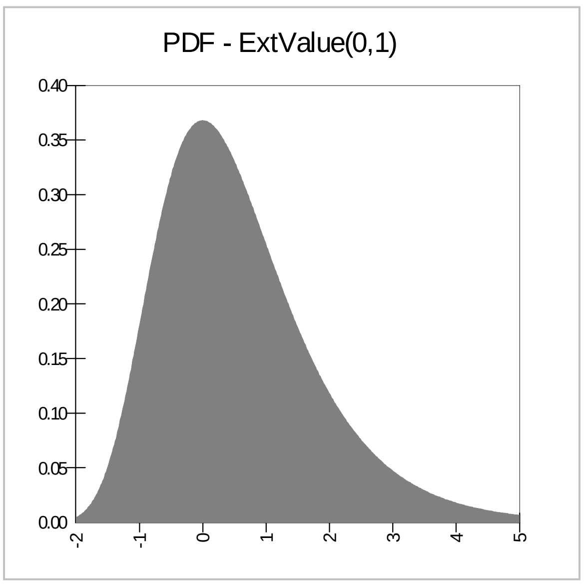
1.139547

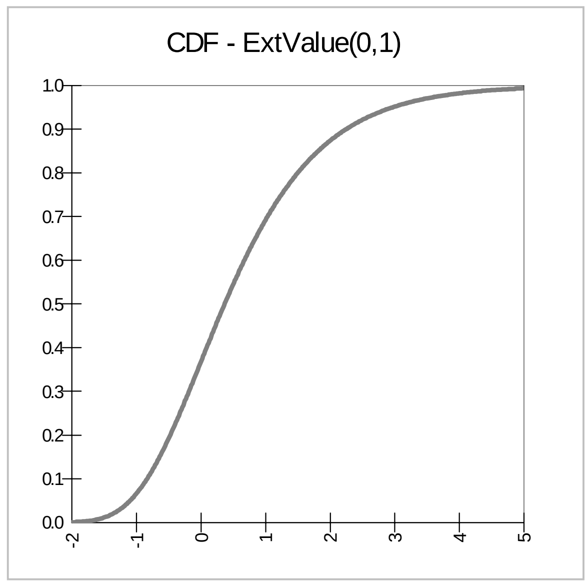
Kurtosis:

5.4

Mode:

a





Gamma

RISKGamma(α, β)

Parameters:

α	continuous shape parameter	$\alpha > 0$
β	continuous scale parameter	$\beta > 0$

Domain:

$0 < x < +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{\beta\Gamma(\alpha)}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta}$$

$$F(x) = \frac{\Gamma_{x/\beta}(\alpha)}{\Gamma(\alpha)}$$

where Γ is the *Gamma Function* and Γ_x is the *Incomplete Gamma Function*.

Mean:

$$\beta\alpha$$

Variance:

$$\beta^2\alpha$$

Skewness:

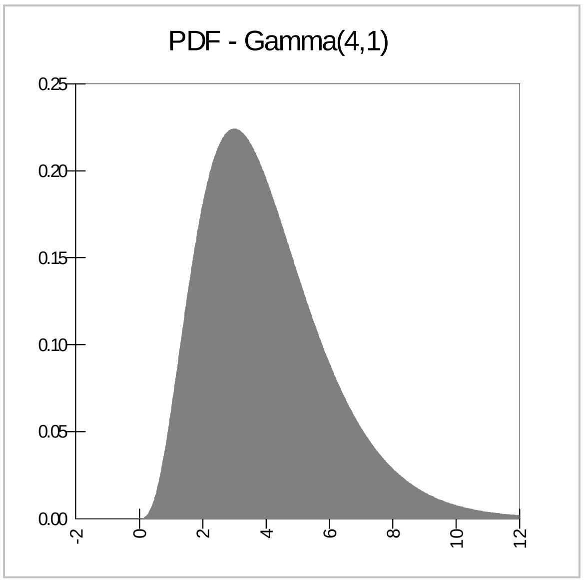
$$\frac{2}{\sqrt{\alpha}}$$

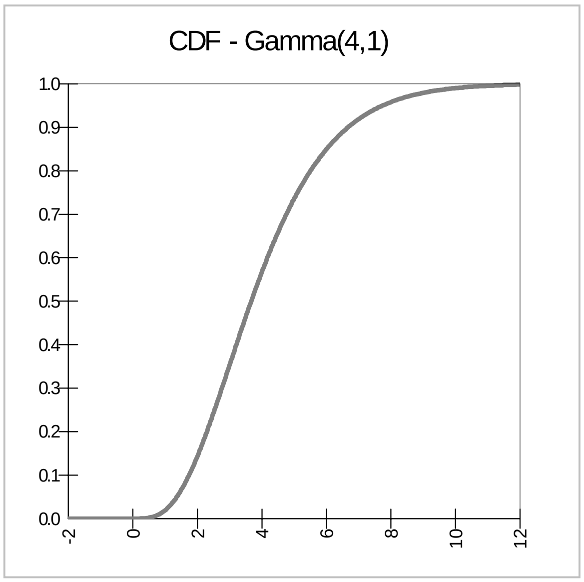
Kurtosis:

$$3+\frac{6}{\alpha}$$

Mode:

$\beta(\alpha-1)$	if $\alpha \geq 1$
Not Defined	if $\alpha < 1$





General

RISKGeneral(min, max, {x}, {p})

Parameters:

min	continuous parameter	$min < max$
max	continuous parameter	
$\mathbf{\{x\} = \{x_1, x_2, \ldots, x_N\}}$	array of continuous parameters	$min \le x_i \le max$
$\mathbf{\{p\} = \{p_1, p_2, \ldots, p_N\}}$	array of continuous parameters	$p_i \ge 0$

Domain:

$min \le x \le max$ continuous

Density and Cumulative Functions:

$$f(x) = p_i + \left[\frac{x - x_i}{x_{i+1} - x_i}\right](p_{i+1} - p_i) \quad \text{for } x_i \le x \le x_{i+1}$$

$$F(x) = F(x_i) + (x - x_i)\left[p_i + \frac{(p_{i+1} - p_i)(x - x_i)}{2(x_{i+1} - x_i)}\right] \quad \text{for } x_i \le x \le x_{i+1}$$

With the assumptions:

1. The arrays are ordered from left to right
2. The {p} array has been normalized to give the general distribution unit area.
3. The *i* index runs from 0 to N+1, with two extra elements : $x_0 \equiv min$, $p_0 \equiv 0$ and $x_{N+1} \equiv max$, $p_{N+1} \equiv 0$.

Mean:

No Closed Form

Variance:

No Closed Form

Skewness:

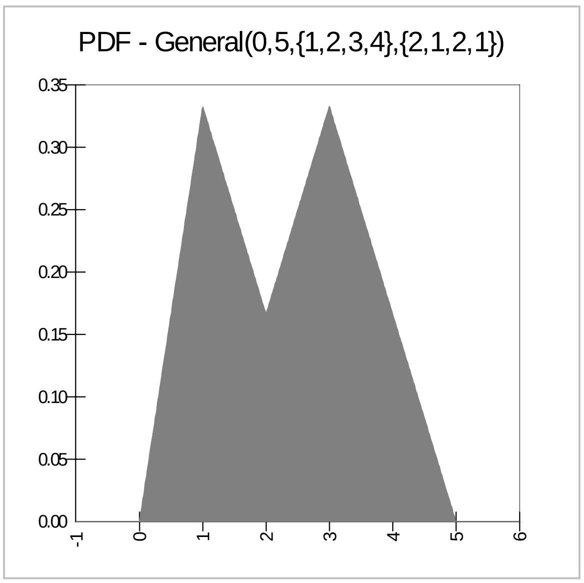
No Closed Form

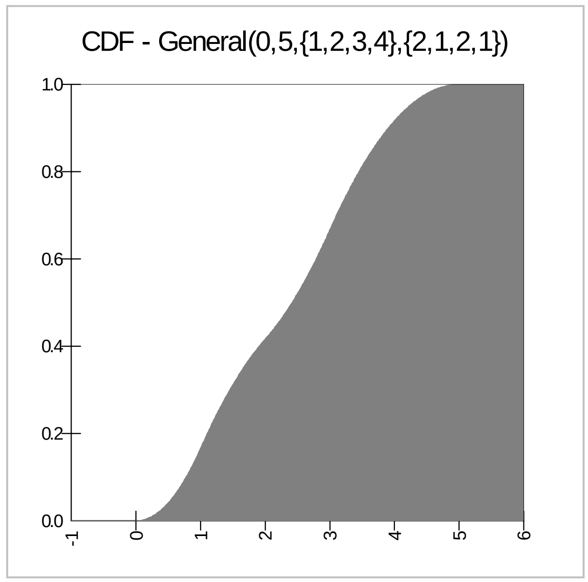
Kurtosis:

No Closed Form

Mode:

No Closed Form





Geometric

RISKGeomet(p)

Parameters:

p	continuous "success" probability	$0 < p < 1$

Domain:

$0 \le x < +\infty$ discrete

Mass and Cumulative Functions:

$$f(x) = p(1-p)^x$$

$$F(x) = 1 - (1-p)^{x+1}$$

Mean:

$$\frac{1}{p} - 1$$

Variance:

$$\frac{1-p}{p^2}$$

<u>Skewness</u>:

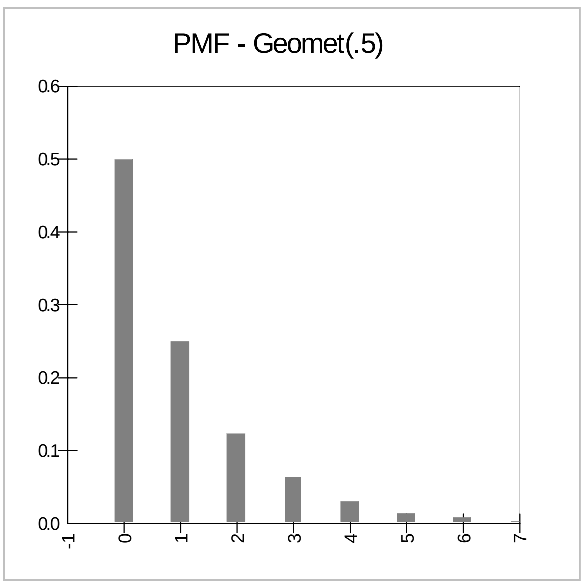
$$\frac{(2-p)}{\sqrt{1-p}}$$

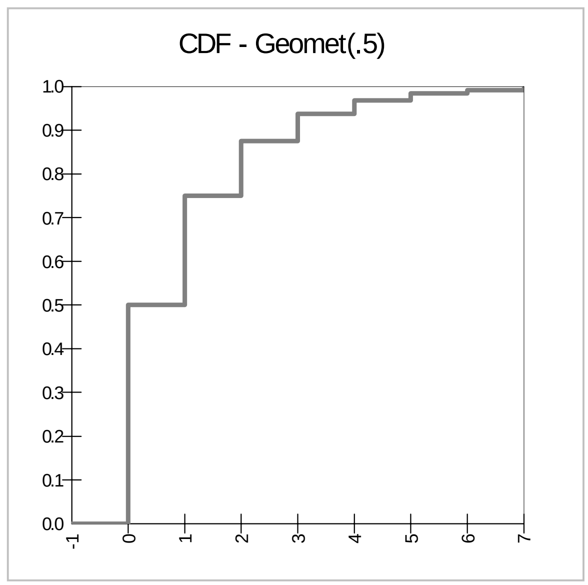
<u>Kurtosis</u>:

$$9+\frac{p}{(1-p)^2}$$

<u>Mode</u>:

0





Histogram

RISKHistogrm(min, max, {p})

Parameters:

min	continuous parameter	min < max
max	continuous parameter	
{p} = {$p_1, p_2, \ldots, p_N$}	array of continuous parameters	$p_i \geq 0$

Domain:

$\min \leq x \leq \max$ continuous

Density and Cumulative Functions:

$$f(x) = p_i \qquad \text{for } x_i \leq x < x_{i+1}$$

$$F(x) = F(x_i) + p_i\left(\frac{x - x_i}{x_{i+1} - x_i}\right) \qquad \text{for } x_i \leq x \leq x_{i+1}$$

$$\text{where } x_i \equiv \min + i\left(\frac{\max - \min}{N}\right)$$

The {p} array has been normalized to give the histogram unit area.

Mean:

No Closed Form

Variance:

No Closed Form

Skewness:

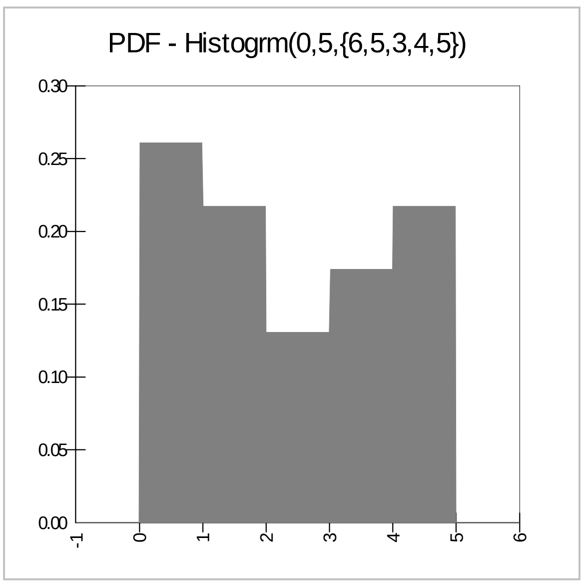
No Closed Form

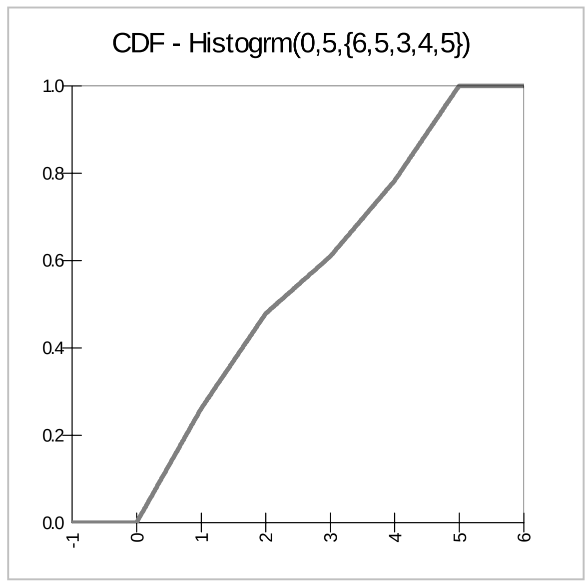
Kurtosis:

No Closed Form

Mode:

Not Uniquely Defined.





Hypergeometric

RISKHyperGeo(n, D, M)

Parameters:

n	*the number of draws*	integer	$0 < n < M$
D	*the number of "tagged" items*	integer	$0 < D < M$
M	*the total number of items*	integer	$M > 0$

Domain:

$\max(0,n+D-M) \le x \le \min(n,D)$ discrete

Mass and Cumulative Functions:

$$f(x) = \frac{\binom{D}{x}\binom{M-D}{n-x}}{\binom{M}{n}}$$

$$F(x) = \sum_{i=1}^{x} \frac{\binom{D}{x}\binom{M-D}{n-x}}{\binom{M}{n}}$$

Mean:

$$\frac{nD}{M}$$

Variance:

$$\frac{nD}{M^2}\left[\frac{(M-D)(M-n)}{(M-1)}\right] \qquad \text{for } M>1$$

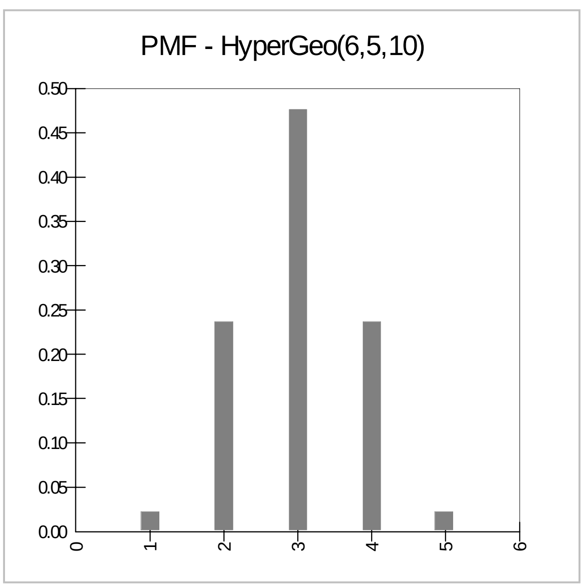
Skewness:

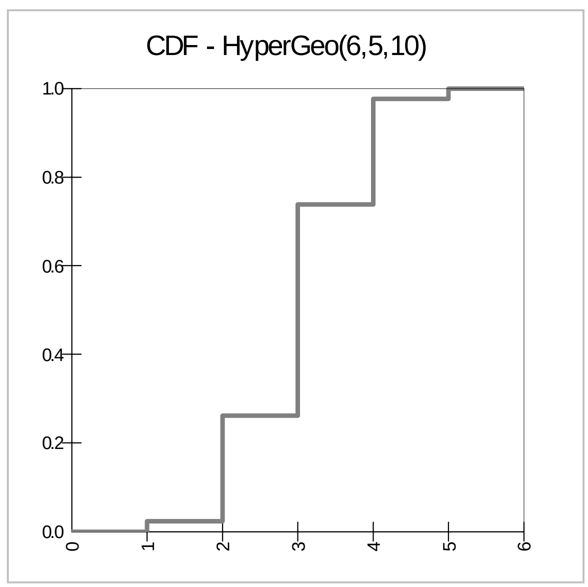
$$\frac{(M-2D)(M-2n)}{M-2}\sqrt{\frac{M-1}{nD(M-D)(M-n)}} \qquad \text{for } M>2$$

Mode:

(bimodal)	x_m and x_m-1	if x_m is integral
(unimodal)	biggest integer less than x_m	otherwise

where $x_m \equiv \frac{(n+1)(D+1)}{M+2}$





Integer Uniform

RISKIntUniform(min, max)

Parameters:

min	discrete boundary parameter	min < max
max	discrete boundary probability	

Domain:

$\min \le x \le \max$ discrete

Mass and Cumulative Functions:

$$f(x) = \frac{1}{\max - \min + 1}$$

$$F(x) = \frac{x - \min + 1}{\max - \min + 1}$$

Mean:

$$\frac{\min + \max}{2}$$

Variance:

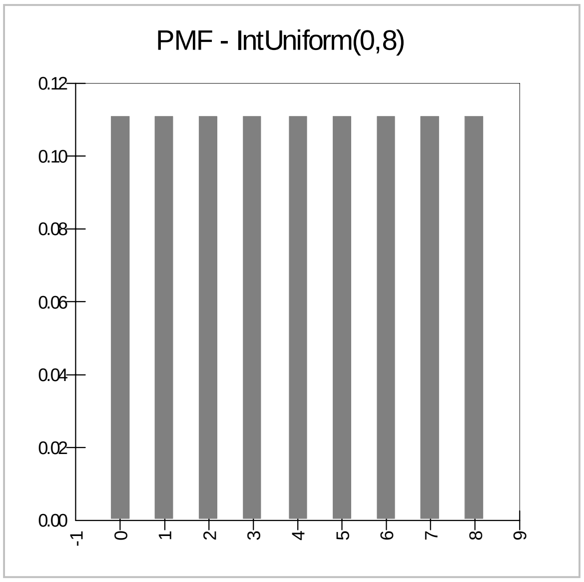
$$\left(\frac{\max - \min}{6}\right)\left(\frac{\max - \min}{2} + 1\right)$$

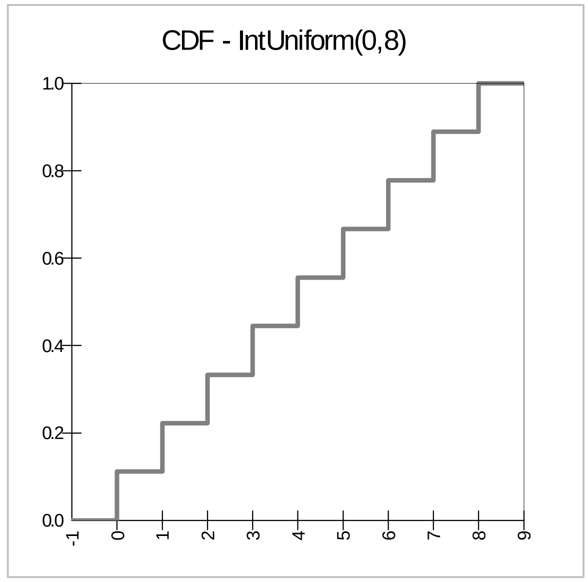
Skewness:

0

Mode:

Not uniquely defined





Inverse Gaussian

RISKInvGauss(μ, λ)

Parameters:

μ	continuous parameter	$\mu > 0$
λ	continous parameter	$\lambda > 0$

Domain:

$x > 0$ continuous

Density and Cumulative Functions:

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\; e^{-\left[\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]}$$

$$F(x) = \Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right] + e^{2\lambda/\mu}\,\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\right]$$

where Φ is the *Error Function*.

Mean:

μ

Variance:

$$\frac{\mu^3}{\lambda}$$

Skewness:

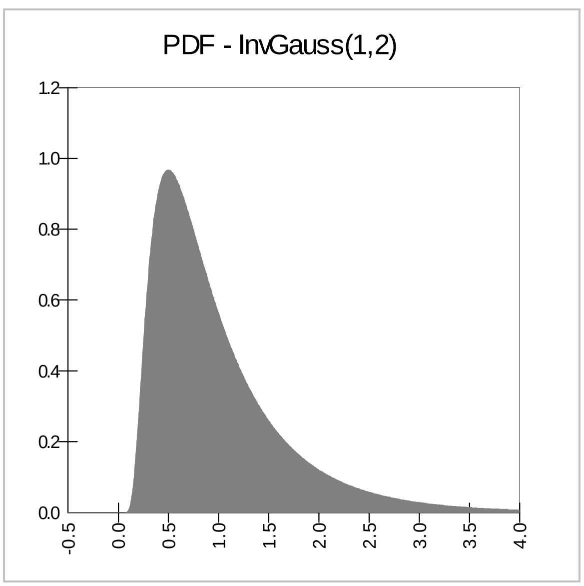
$$3\sqrt{\frac{\mu}{\lambda}}$$

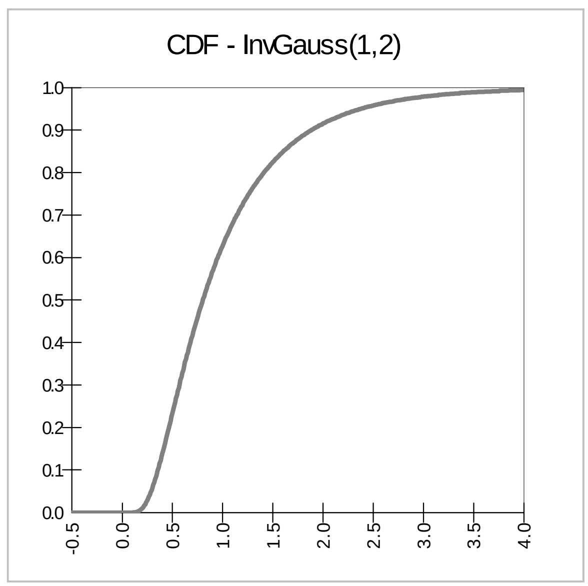
Kurtosis:

$$3+15\frac{\mu}{\lambda}$$

Mode:

$$\mu\left[\left(1+\frac{9\mu^2}{4\lambda^2}\right)-\frac{3\mu}{2\lambda}\right]$$





Logistic

RISKLogistic(α, β)

Parameters:

α	continuous location parameter	
β	continuous scale parameter	$\beta > 0$

Domain:

$-\infty \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{\operatorname{sech}^2\left(\frac{1}{2}\left(\frac{x-\alpha}{\beta}\right)\right)}{4\beta}$$

$$F(x) = \frac{1+\tanh\left(\frac{1}{2}\left(\frac{x-\alpha}{\beta}\right)\right)}{2}$$

where “sech” is the *Hyperbolic Secant Function* and “tanh” is the *Hyperbolic Tangent Function*.

Mean:

α

Variance:

$$\frac{\pi^2\beta^2}{3}$$

Skewness:

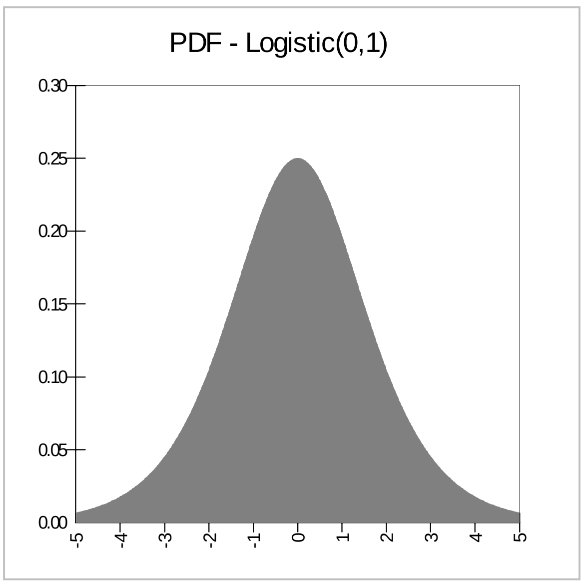
0

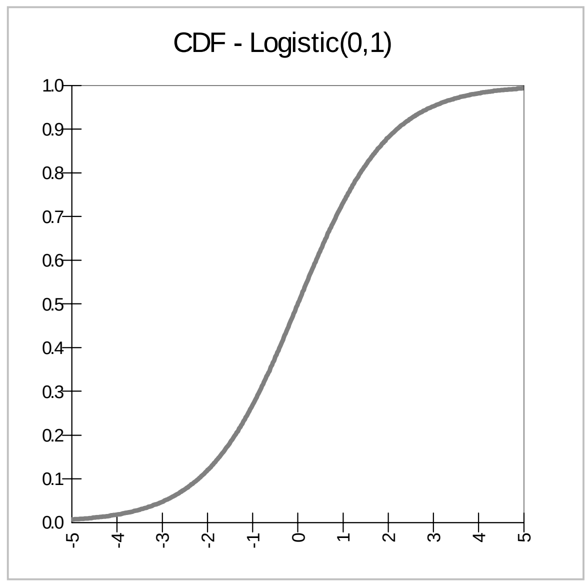
Kurtosis:

4.2

Mode:

α





Log-Logistic

RISKLogLogistic(γ, β, α)

Parameters:

γ	continuous location parameter	
β	continous scale parameter	$\beta > 0$
α	continuous shape parameter	$\alpha > 0$

Definitions:

$$\theta \equiv \frac{\pi}{\alpha}$$

Domain:

$\gamma \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{\alpha\, t^{\alpha-1}}{\beta\left(1+t^{\alpha}\right)^2}$$

$$F(x) = \frac{1}{1+\left(\frac{1}{t}\right)^{\alpha}}$$

with $t \equiv \frac{x-\gamma}{\beta}$

Mean:

$\beta\theta\,\csc(\theta)+\gamma$ for $\alpha > 1$

Variance:

$$\beta^2\theta\left[2\csc(2\theta)-\theta\csc^2(\theta)\right]$$ for $\alpha > 2$

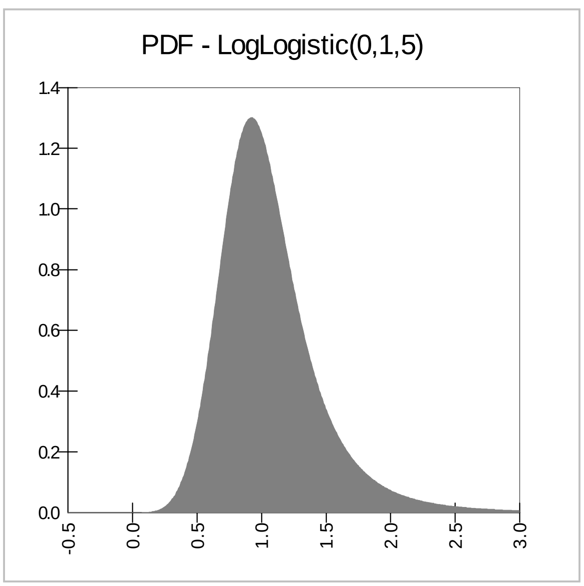
Skewness:

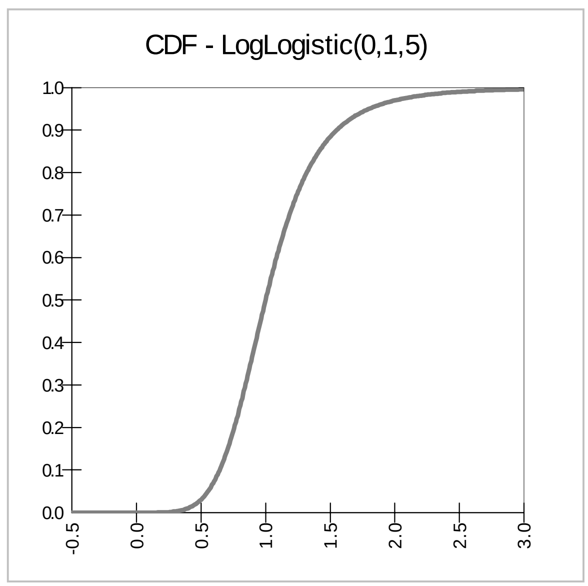
$$\frac{3\csc(3\theta)-6\theta\csc(2\theta)\csc(\theta)+2\theta^2\csc^3(\theta)}{\sqrt{\theta}\left[2\csc(2\theta)-\theta\csc^2(\theta)\right]^{3/2}}$$ for $\alpha > 3$

Mode:

$$\gamma+\beta\left[\frac{\alpha-1}{\alpha+1}\right]^{1/\alpha}$$ for $\alpha > 1$

$$0$$ for $\alpha \le 1$





Lognormal (Format 1)

RISKLognorm(μ, σ)

Parameters:

μ	continuous parameter	$\mu > 0$
σ	continous parameter	$\sigma > 0$

Domain:

$0 \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{x\sqrt{2\pi}\sigma'} e^{-\frac{1}{2}\left[\frac{\ln x - \mu'}{\sigma'}\right]^2}$$

$$F(x) = \Phi\left(\frac{\ln x - \mu'}{\sigma'}\right)$$

with $\mu' \equiv \ln\left[\frac{\mu^2}{\sqrt{\sigma^2 + \mu^2}}\right]$ and $\sigma' \equiv \sqrt{\ln\left[1 + \left(\frac{\sigma}{\mu}\right)^2\right]}$

where Φ is the *Error Function*.

Mean:

μ

Variance:

σ^2

Skewness:

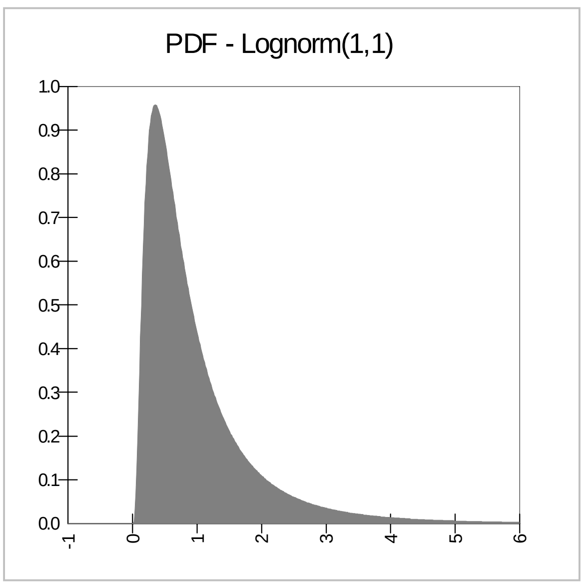
$$\left(\frac{\sigma}{\mu}\right)^3 + 3\left(\frac{\sigma}{\mu}\right)$$

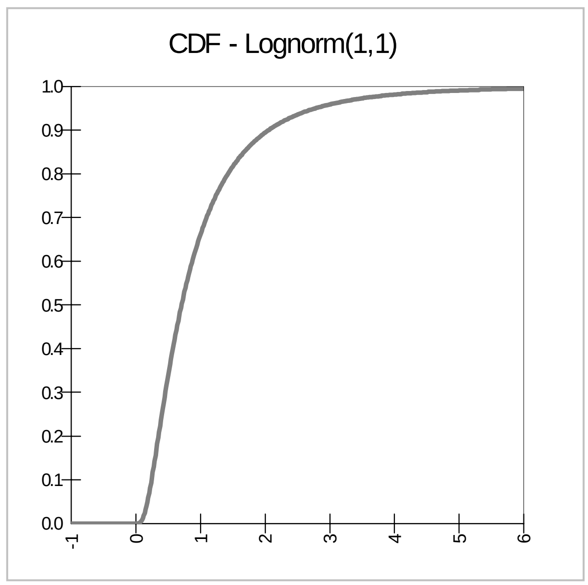
Kurtosis:

$$\omega^4 + 2\omega^3 + 3\omega^2 - 3 \qquad \text{with } \omega \equiv 1 + \left(\frac{\sigma}{\mu}\right)$$

Mode:

$$\frac{\mu^4}{\left(\sigma^2 + \mu^2\right)^{3/2}}$$





Lognormal (Format 2)

RISKLognorm2(μ, σ)

Parameters:

μ	continuous parameter	
σ	continous parameter	$\sigma > 0$

Domain:

$0 \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left[\frac{\ln x - \mu}{\sigma}\right]^2}$$

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

where Φ is the *Error Function*.

Mean:

$$e^{\mu + \frac{\sigma^2}{2}}$$

Variance:

$$e^{2\mu}\omega(\omega - 1)$$ with $\omega \equiv e^{\sigma^2}$

Skewness:

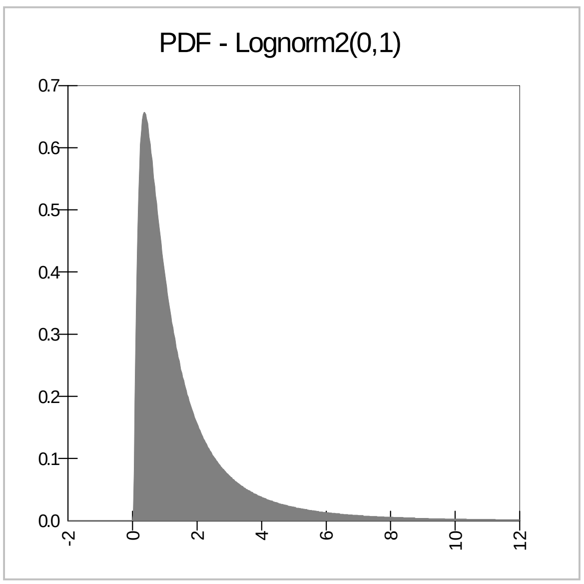
$$(\omega+2)\sqrt{\omega-1} \qquad \text{with } \omega \equiv e^{\sigma^2}$$

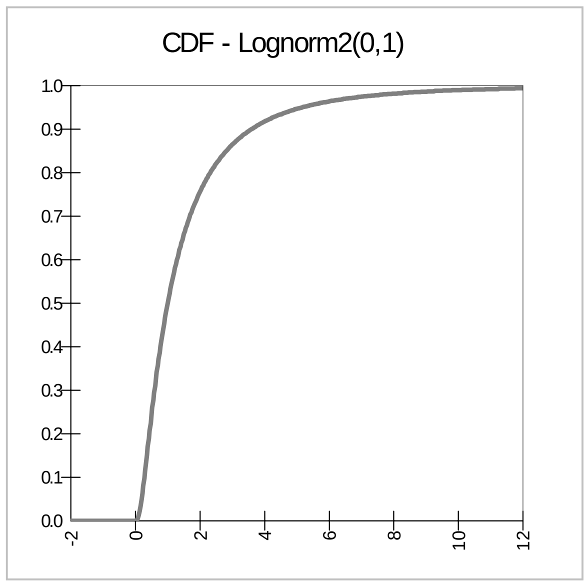
Kurtosis:

$$\omega^4 + 2\omega^3 + 3\omega^2 - 3 \qquad \text{with } \omega \equiv e^{\sigma^2}$$

Mode:

$$e^{\mu-\sigma^2}$$





Negative Binomial

RISKNegBin(s, p)

Parameters:

s	*the number of successes*	discrete parameter	$s > 0$
p	*probability of a single success*	continuous parameter	$0 < p < 1$

Domain:

$0 \le x \le n$ discrete

Density and Cumulative Functions:

$$f(x) = \binom{s+x-1}{x} p^s (1-p)^x$$

$$F(x) = p^s \sum_{i=0}^{x} \binom{s+i-1}{i} (1-p)^i$$

Where ( ) is the *Binomial Coefficient.*

Mean:

$$\frac{s(1-p)}{p}$$

Variance:

$$\frac{s(1-p)}{p^2}$$

Skewness:

$$\frac{2-p}{\sqrt{s(1-p)}}$$

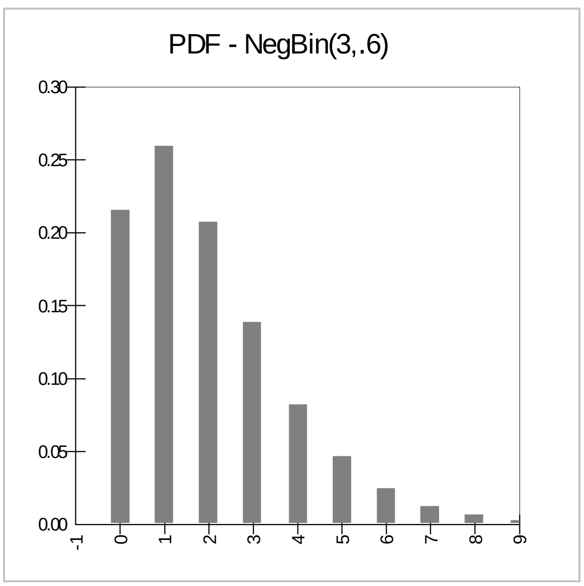
Kurtosis:

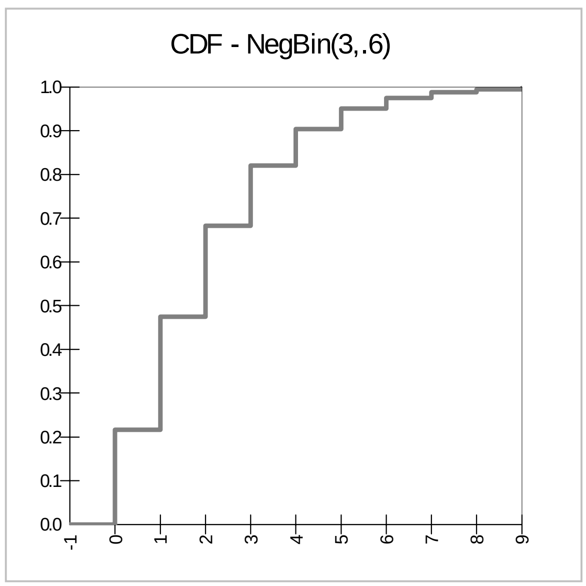
$$3+\frac{6}{s}+\frac{p^2}{s(1-p)}$$

Mode:

(bimodal)	z and z + 1	integer z > 0
(unimodal)	0	z < 0
(unimodal)	smallest integer greater than z	otherwise

where $z \equiv \frac{s(1-p)-1}{p}$





Normal

RISKNormal(μ, σ)

Parameters:

μ	continuous location parameter	
σ	continuous scale parameter	$\sigma > 0$

Domain:

$-\infty \leq x \leq +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

where Φ is the *Error Function*.

Mean:

μ

Variance:

σ^2

Skewness:

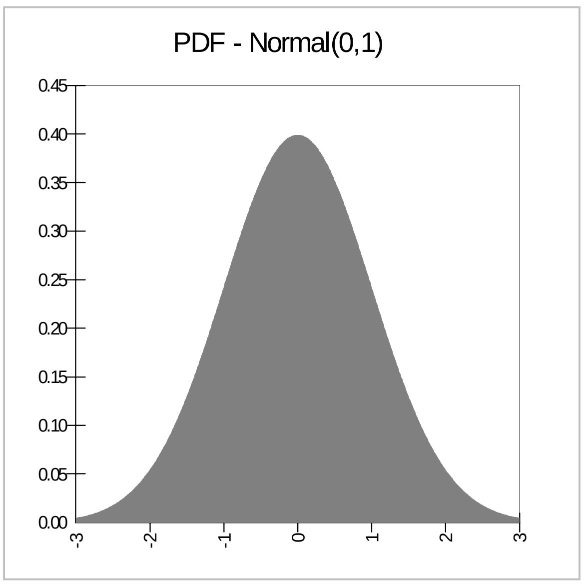
0

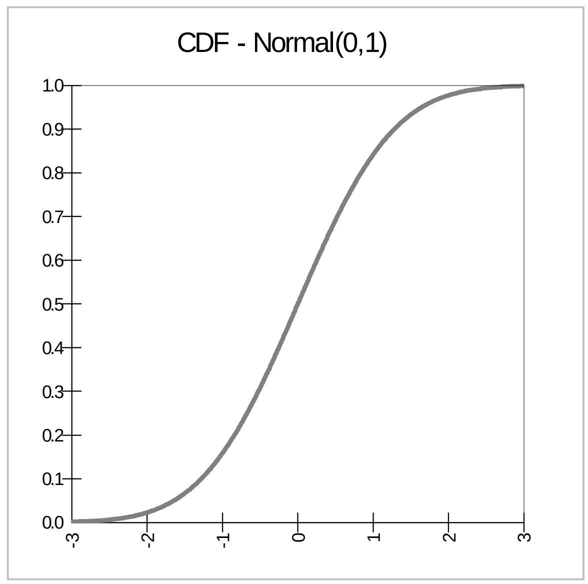
Kurtosis:

3

Mode:

μ





Pareto (First Kind)

Pareto(θ, a)

Parameters:

θ	continuous shape parameter	$\theta > 0$
a	continuous scale parameter	$a > 0$

Domain:

$a \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{\theta a^{\theta}}{x^{\theta+1}}$$

$$F(x) = 1 - \left(\frac{a}{x}\right)^{\theta}$$

Mean:

$$\frac{a\theta}{\theta - 1}$$ for $\theta > 1$

Variance:

$$\frac{\theta a^2}{(\theta - 1)^2 (\theta - 2)}$$ for $\theta > 2$

Skewness:

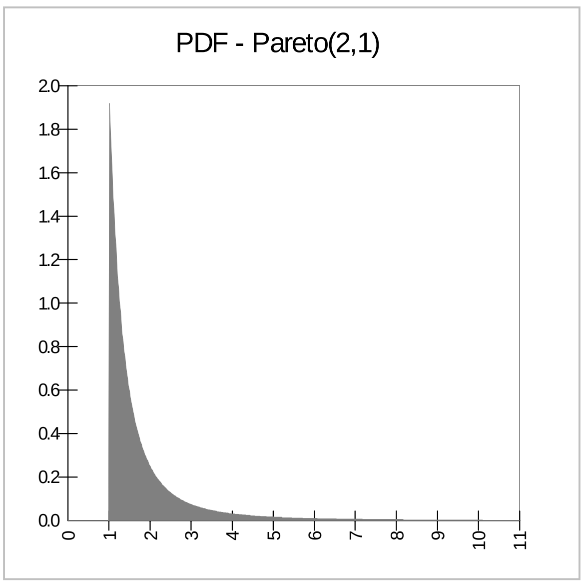
$$2\frac{\theta+1}{\theta-3}\sqrt{\frac{\theta-2}{\theta}}$$ for $\theta > 3$

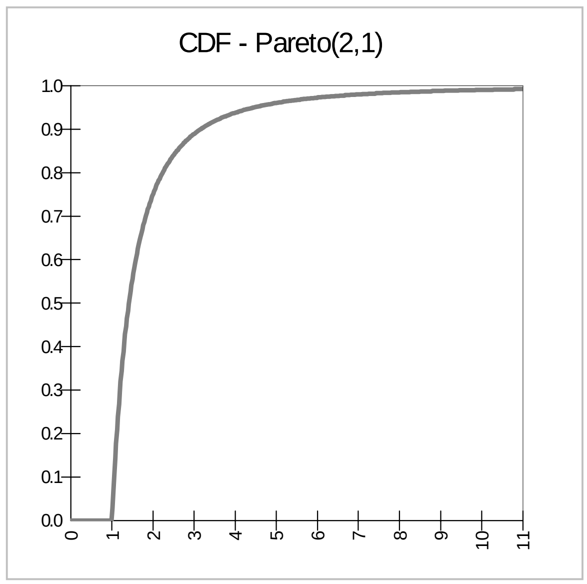
Kurtosis:

$$\frac{3(\theta-2)\left(3\theta^2+\theta+2\right)}{\theta(\theta-3)(\theta-4)}$$ for $\theta > 4$

Mode:

a





Pareto (Second Kind)

RISKPareto2(b, q)

Parameters:

b	continuous scale parameter	$b > 0$
q	continuous shape parameter	$q > 0$

Domain:

$0 \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{qb^q}{(x+b)^{q+1}}$$

$$F(x) = 1 - \frac{b^q}{(x+b)^q}$$

Mean:

$$\frac{b}{q-1}$$ for $q > 1$

Variance:

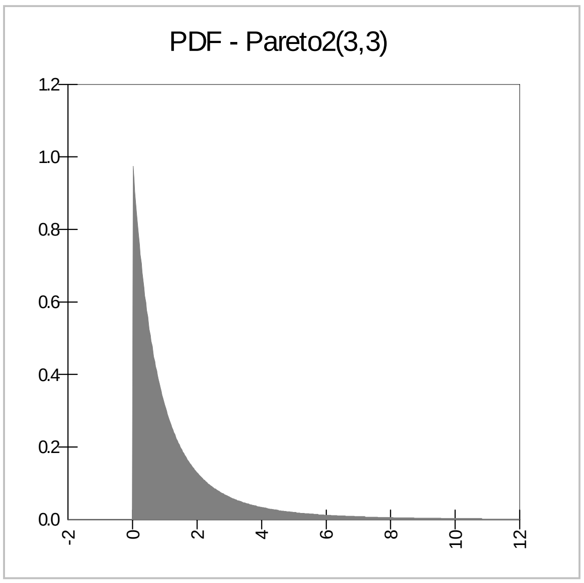
$$\frac{b^2 q}{(q-1)^2 (q-2)}$$ for $q > 2$

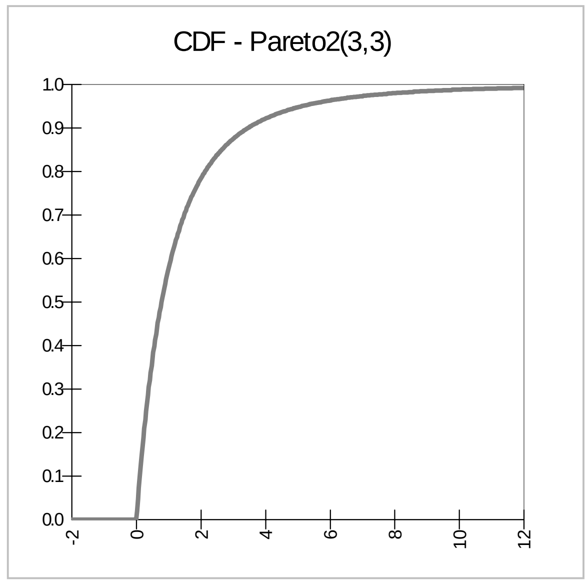
Skewness:

$$2\sqrt{\frac{q-2}{q}}\left[\frac{q+1}{q-3}\right] \qquad \text{for } q > 3$$

Mode:

0





Pearson Type V

RISKPearson5(α, β)

Parameters:		
α	continuous shape parameter	$\alpha > 0$
β	continous scale parameter	$\beta > 0$

Domain:

$0 \le x < +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{\beta\Gamma(\alpha)} \cdot \frac{e^{-\beta/x}}{(x/\beta)^{\alpha+1}}$$

F(x) Has No Closed Form

Mean:

$$\frac{\beta}{\alpha - 1}$$ for $\alpha > 1$

Variance:

$$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$$ for $\alpha > 2$

Skewness:

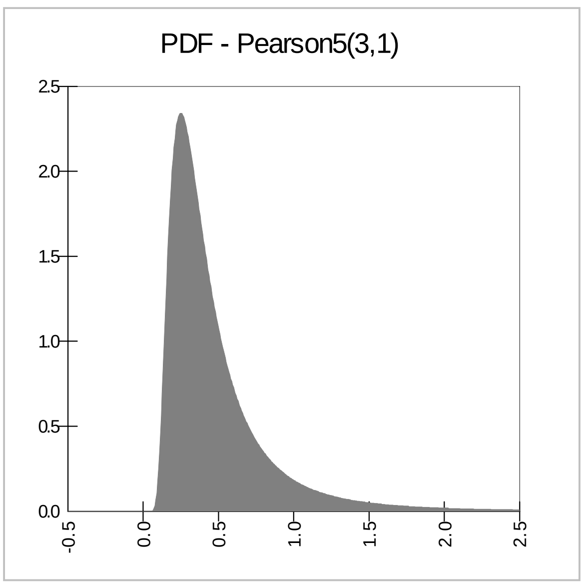
$$\frac{4\sqrt{\alpha-2}}{\alpha-3} \qquad \text{for } \alpha > 3$$

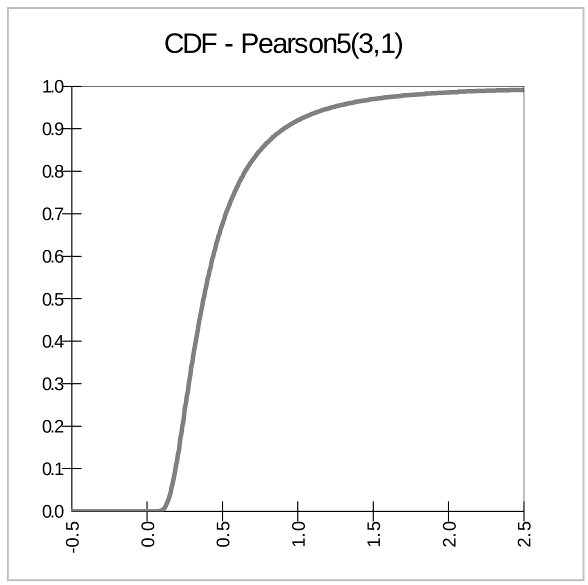
Kurtosis:

$$\frac{3(\alpha+5)(\alpha-2)}{(\alpha-3)(\alpha-4)} \qquad \text{for } \alpha > 4$$

Mode:

$$\frac{\beta}{\alpha+1}$$





Pearson Type VI

RISKPearson6(α_1, α_2, β)

Parameters:

α_1	continuous shape parameter	$\alpha_1 > 0$
α_2	continuous shape parameter	$\alpha_2 > 0$
β	continuous scale parameter	$\beta > 0$

Domain:

$0 \le x < +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{\beta B(\alpha_1, \alpha_2)} \times \frac{(x/\beta)^{\alpha_1 - 1}}{\left(1 + \frac{x}{\beta}\right)^{\alpha_1 + \alpha_2}}$$

F(x) Has No Closed Form.

where B is the *Beta Function*.

Mean:

$$\frac{\beta\alpha_1}{\alpha_2 - 1}$$ for $\alpha_2 > 1$

Variance:

$$\frac{\beta^2\alpha_1(\alpha_1 + \alpha_2 - 1)}{(\alpha_2 - 1)^2(\alpha_2 - 2)}$$ for $\alpha_2 > 2$

Skewness:

$$2\sqrt{\frac{\alpha_2 - 2}{\alpha_1(\alpha_1+\alpha_2-1)}}\left[\frac{2\alpha_1+\alpha_2-1}{\alpha_2-3}\right]$$ for $\alpha_2 > 3$

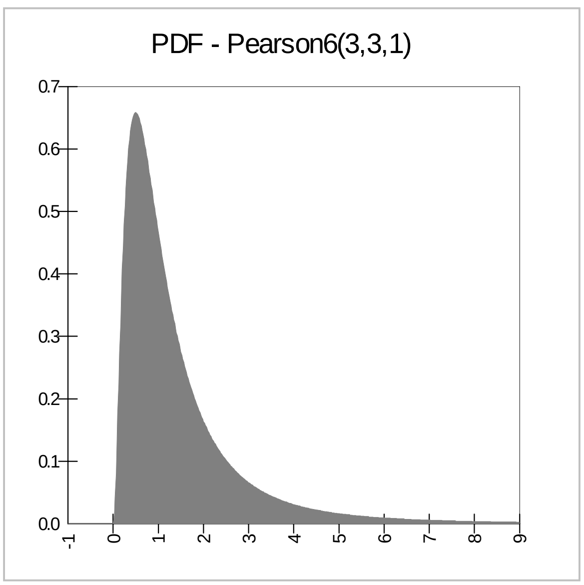
Kurtosis:

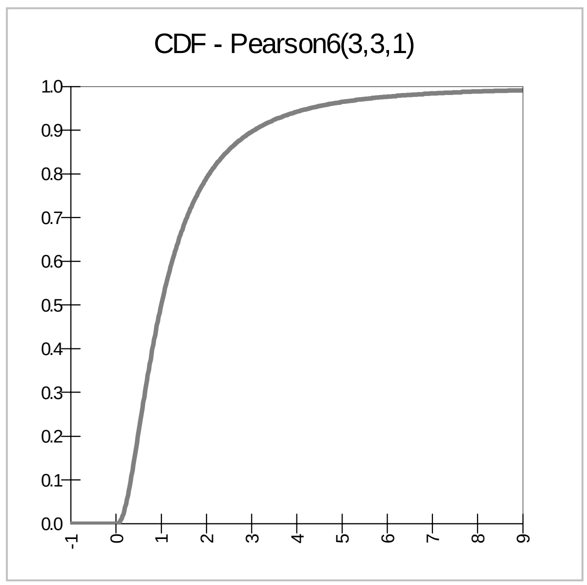
$$\frac{3(\alpha_2-2)}{(\alpha_2-3)(\alpha_2-4)}\left[\frac{2(\alpha_2-1)^2}{\alpha_1(\alpha_1+\alpha_2-1)}+(\alpha_2+5)\right]$$ for $\alpha_2 > 4$

Mode:

$$\frac{\beta(\alpha_1-1)}{\alpha_2+1}$$ for $\alpha_1 > 1$

0 otherwise





Pert (Beta)

RISKPert(min, m.likely, max)

Definitions:

$$\mu \equiv \frac{\text{min} + 4 \cdot \text{m.likely} + \text{max}}{6} \qquad \alpha_1 \equiv 6\left[\frac{\mu - \text{min}}{\text{max} - \text{min}}\right] \qquad \alpha_2 \equiv 6\left[\frac{\text{max} - \mu}{\text{max} - \text{min}}\right]$$

Parameters:

min	continuous boundary parameter	min < max
m.likely	continuous parameter	min < m.likely < max
max	continuous boundary parameter	

Domain:

$\text{min} \le x \le \text{max}$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{(x - \text{min})^{\alpha_1 - 1}(\text{max} - x)^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)(\text{max} - \text{min})^{\alpha_1 + \alpha_2 - 1}}$$

$$F(x) = \frac{B_z(\alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \equiv I_z(\alpha_1, \alpha_2) \qquad \text{with} \quad z \equiv \frac{x - \text{min}}{\text{max} - \text{min}}$$

where B is the *Beta Function* and B_z is the *Incomplete Beta Function*.

Mean:

$$\mu \equiv \frac{\text{min} + 4 \cdot \text{m.likely} + \text{max}}{6}$$

Variance:

$$\frac{(\mu - \min)(\max - \mu)}{7}$$

Skewness:

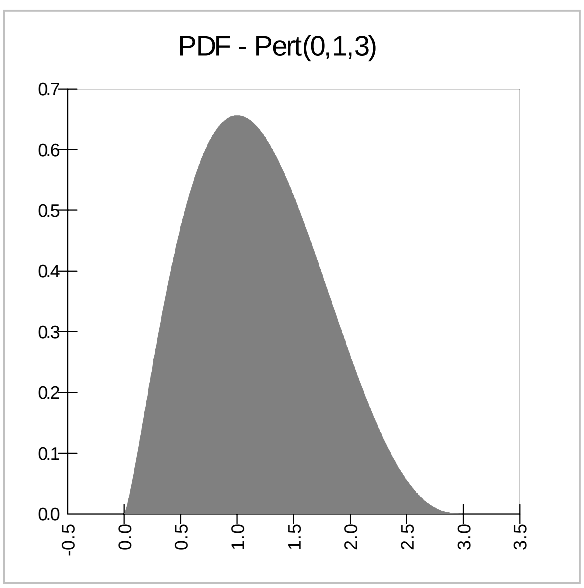
$$\frac{\min + \max - 2\mu}{4}\sqrt{\frac{7}{(\mu - \min)(\max - \mu)}}$$

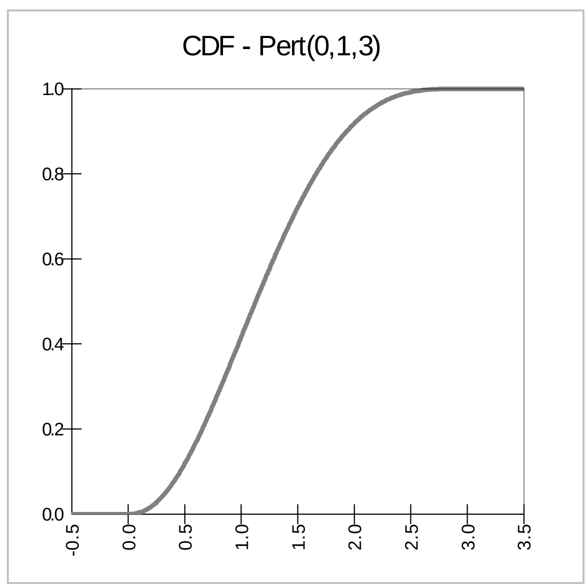
Kurtosis:

$$3\frac{(\alpha_1 + \alpha_2 + 1)\left(2(\alpha_1 + \alpha_2)^2 + \alpha_1\alpha_2(\alpha_1 + \alpha_2 - 6)\right)}{\alpha_1\alpha_2(\alpha_1 + \alpha_2 + 2)(\alpha_1 + \alpha_2 + 3)}$$

Mode:

m.likely





Poisson

RISKPoisson(λ)

Parameters:

λ	*mean number of successes*	continuous	$\lambda > 0$

Domain:

$0 \le x < +\infty$ discrete

Mass and Cumulative Functions:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$F(x) = e^{-\lambda} \sum_{n=0}^{x} \frac{\lambda^n}{n!}$$

Mean:

λ

Variance:

λ

Skewness:

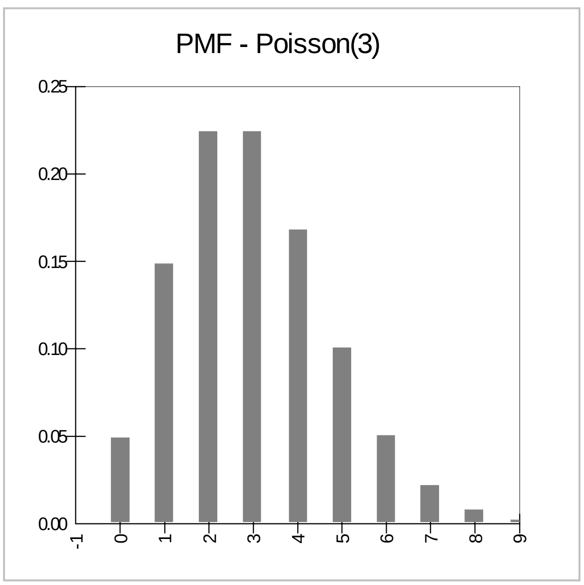
$$\frac{1}{\sqrt{\lambda}}$$

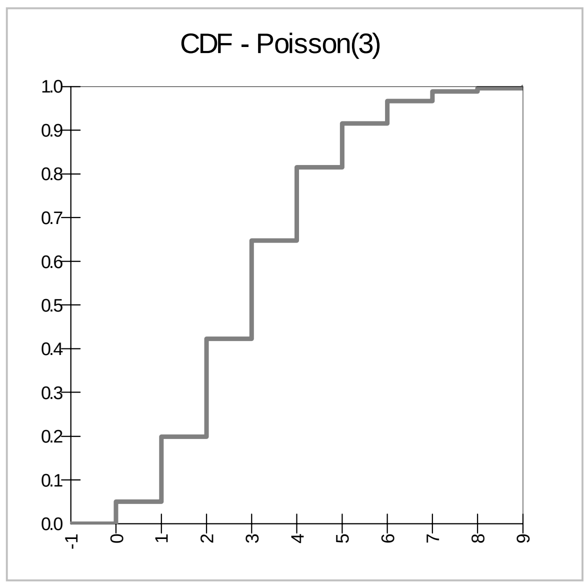
Kurtosis:

$$3+\frac{1}{\lambda}$$

Mode:

(bimodal)	λ and λ+1 (bimodal)	if λ is an integer
(unimodal)	largest integer less than λ	otherwise





Rayleigh

RISKRayleigh(b)

Parameters:

b	continuous scale parameter	$b > 0$

Domain:

$0 \le x < +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{x}{b^2} e^{-\frac{1}{2}\left(\frac{x}{b}\right)^2}$$

$$F(x) = 1 - e^{-\frac{1}{2}\left(\frac{x}{b}\right)^2}$$

Mean:

$$b\sqrt{\frac{\pi}{2}}$$

Variance:

$$b^2\left(2 - \frac{\pi}{2}\right)$$

Skewness:

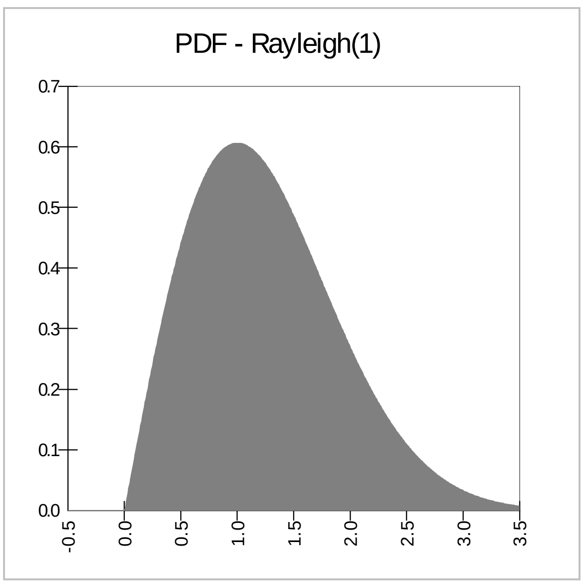
$$\frac{2(\pi-3)\sqrt{\pi}}{(4-\pi)^{3/2}} \approx 0.6311$$

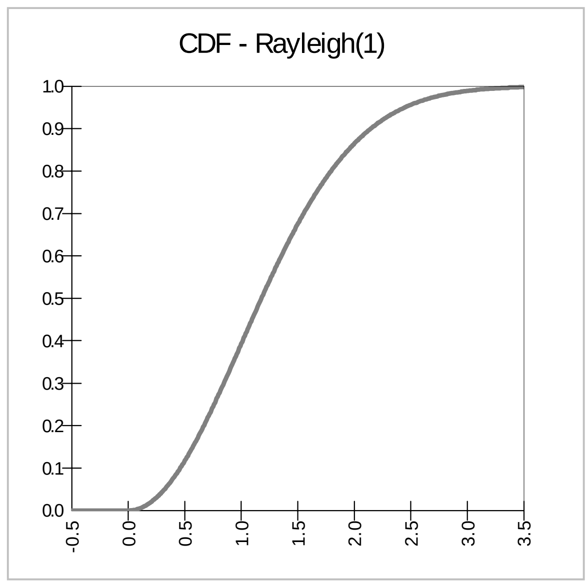
Kurtosis:

$$\frac{32-3\pi^2}{(4-\pi)^2} \approx 3.2451$$

Mode:

b





Student's "t"

RISKStudent(ν)

Parameters:

ν	*the degrees of freedom*	integer	$\nu > 0$

Domain:

$-\infty \le x \le +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{\sqrt{\pi\nu}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left[\frac{\nu}{\nu + x^2}\right]^{\frac{\nu+1}{2}}$$

$$F(x) = \frac{1}{2}\left[1 + I_s\left(\frac{1}{2}, \frac{\nu}{2}\right)\right] \qquad \text{with } s \equiv \frac{x^2}{\nu + x^2}$$

where Γ is the *Gamma Function* and I_x is the *Incomplete Beta Function*.

Mean:

0 for $\nu > 1$*

*even though the mean is not well defined for $\nu = 1$, the distribution is still symmetrical about 0.

Variance:

$$\frac{\nu}{\nu - 2}$$ for $\nu > 2$

Skewness:

0 for $\nu > 3$*

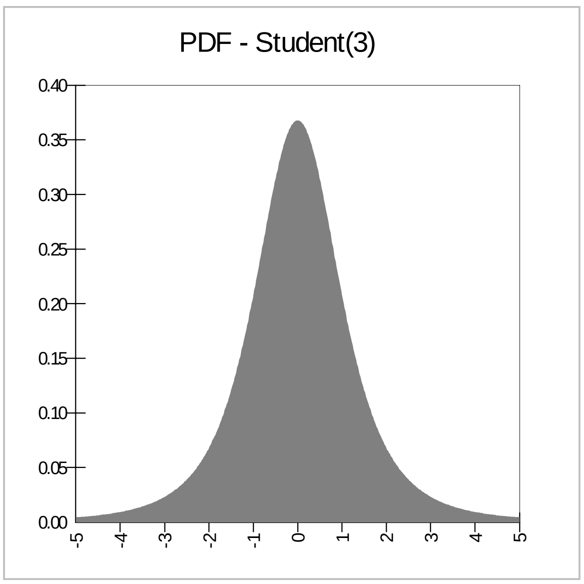
*even though the skewness is not well defined for $\nu \leq 3$, the distribution is still symmetric about 0.

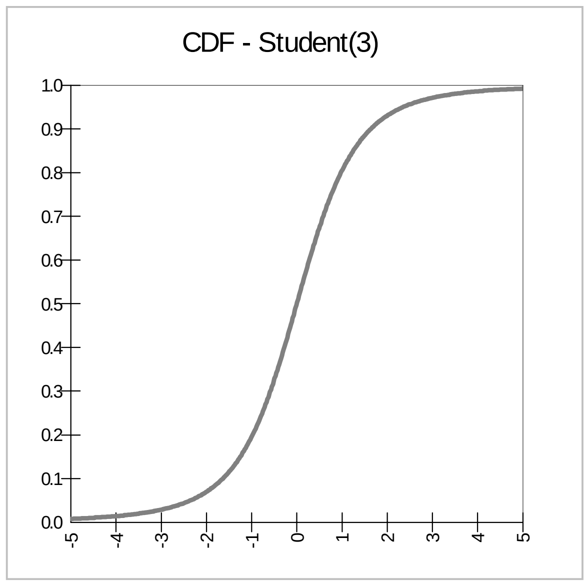
Kurtosis:

$$3\left(\frac{\nu-2}{\nu-4}\right)$$ for $\nu > 4$

Mode:

0





Triangular

RISKTriang(min, m.likely, max)

Parameters:

min	continuous boundary parameter	min < max
m.likely	continuous mode parameter	min ≤ m.likely ≤ max
max	continuous boundary parameter	

Domain:

min ≤ x ≤ max   continuous

Density and Cumulative Functions:

$$f(x) = \frac{2(x - \text{min})}{(\text{m.likely} - \text{min})(\text{max} - \text{min})} \qquad \text{min} \le x \le \text{m.likely}$$

$$f(x) = \frac{2(\text{max} - x)}{(\text{max} - \text{m.likely})(\text{max} - \text{min})} \qquad \text{m.likely} \le x \le \text{max}$$

$$F(x) = \frac{(x - \text{min})^2}{(\text{m.likely} - \text{min})(\text{max} - \text{min})} \qquad \text{min} \le x \le \text{m.likely}$$

$$F(x) = 1 - \frac{(\text{max} - x)^2}{(\text{max} - \text{m.likely})(\text{max} - \text{min})} \qquad \text{m.likely} \le x \le \text{max}$$

Mean:

$$\frac{\text{max} + \text{m.likely} + \text{max}}{3}$$

Variance:

$$\frac{\max^2 + \text{m.likely}^2 + \max^2 - (\max)(\text{m.likely}) - (\text{m.likely})(\min) - (\max)(\min)}{18}$$

Skewness:

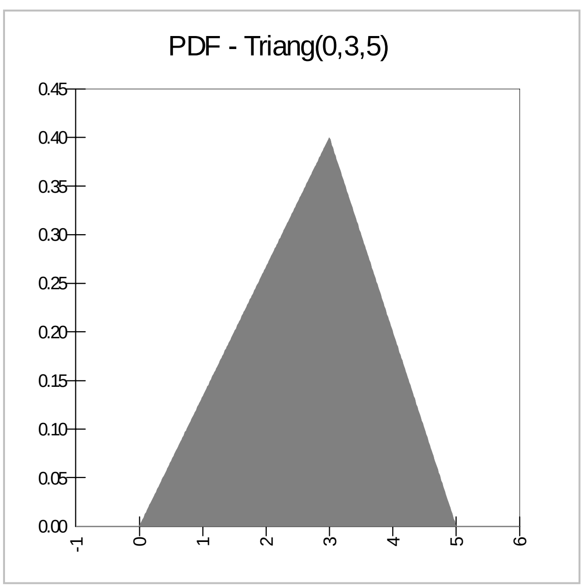
$$\frac{2\sqrt{2}}{5}\frac{f\left(f^2-9\right)}{\left(f^2+3\right)^{3/2}} \qquad \text{where } f \equiv \frac{2(\text{m.likely} - \min)}{\max - \min} - 1$$

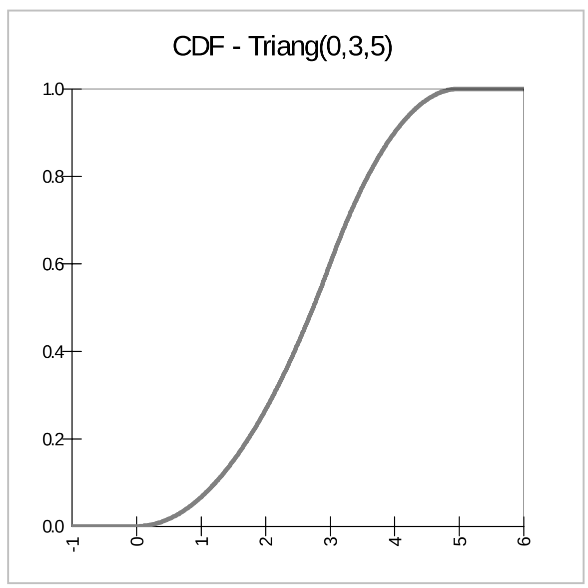
Kurtosis:

2.4

Mode:

m.likely





Uniform

RISKUniform(min, max)

Parameters:

min	continuous boundary parameter	min < max
max	continuous boundary parameter	

Domain:

$\min \le x \le \max$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{1}{\max - \min}$$

$$F(x) = \frac{x - \min}{\max - \min}$$

Mean:

$$\frac{\max - \min}{2}$$

Variance:

$$\frac{(\max - \min)^2}{12}$$

Skewness:

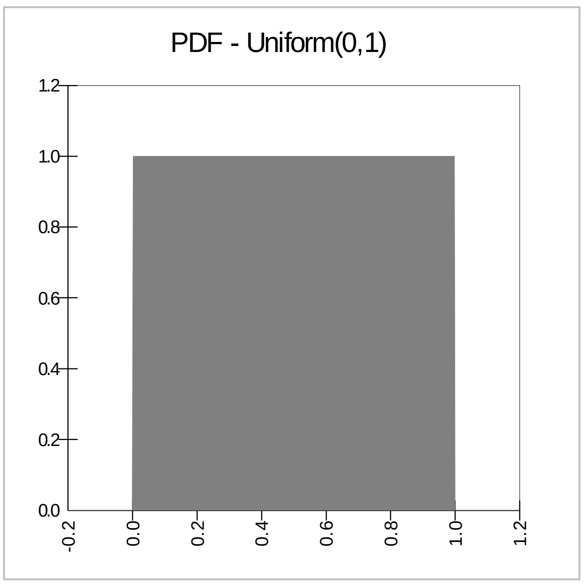
0

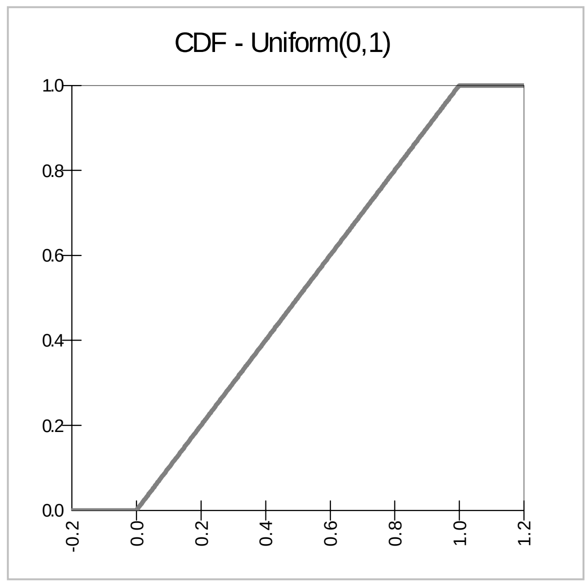
Kurtosis:

1.8

Mode:

Not uniquely defined





Weibull

RISKWeibull(α, β)

Parameters:

α	continuous shape parameter	$\alpha > 0$
β	continuous scale parameter	$\beta > 0$

Domain:

$0 \le x < +\infty$ continuous

Density and Cumulative Functions:

$$f(x) = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}} e^{-(x/\beta)^{\alpha}}$$

$$F(x) = 1 - e^{-(x/\beta)^{\alpha}}$$

Mean:

$$b\Gamma\left(1 + \frac{1}{\alpha}\right)$$

where Γ is the *Gamma Function*.

Variance:

$$\beta^2\left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)\right]$$

where Γ is the *Gamma Function*.

Skewness:

$$\frac{\Gamma\left(1+\frac{3}{\alpha}\right)+3\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)+2\Gamma^3\left(1+\frac{1}{\alpha}\right)}{\left[\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma^2\left(1+\frac{1}{\alpha}\right)\right]^{3/2}}$$

where Γ is the *Gamma Function*.

Mode:

$$\beta\left(1-\frac{1}{\alpha}\right)^{1/\alpha} \qquad \text{for } \alpha > 1$$

$$0 \qquad \text{for } \alpha \le 1$$